# Significant Digits

## Introduction

We are all comfortable with the use of significant digits and we all have an instinctive definition for them. When someone asks your age, you might reply, “I'm 18.” By that response, you specifically mean your age is $18.5 \pm 0.5$ years. Your age is somewhere between 18 and 19 years and you have used two significant digits to express that idea. You could have said, “I'm 18.4036211”, and that would have been a more accurate answer using nine significant digits but that answer would likely be no more helpful than the original.

Loosely defined, significant digits are those digits in a number that are accurate and relevant in a particular situation. We generally employ our ideas about significant digits when we describe a measurement. For example, after measuring the size of a piece of paper with a meter stick, we could say the paper is 27.9 cm long (using three significant digits). We should not say the paper is 27.91324973 cm long because the last six digits are not accurate. That is, they could not be measured accurately with a meter stick.

We are all tempted to use more significant digits than we should. Long strings of digits pour from our calculators and computers and are mysteriously satisfying. That said, we all know that extra significant digits are often misleading. When we brag about our cow, saying “Bossy gave 3.29746238934 gallons of milk today!” we makes ourselves look silly.

## Basic Rules

If we begin with the assumption that we have a number that is both accurate and relevant, we can state a few rules for counting the number of significant digits in the number.

1. All non-zero digits are significant.
2. Zeroes placed at the beginning of a number before other non-zero digits are not significant; 0.032 has two significant digits.
3. Zeroes placed between other digits are always significant; 1004 has four significant digits.
4. Zeroes at the end of a number are significant if the number has a decimal point. There are several variations in the way this rule works. For example, 5.60 has three significant digits. 1230.00 has 6 significant digits and 23400. has five significant digits.
5. Zeros at the end of a number without a decimal point are ambiguous. We cannot tell if they are significant or not; 1230 might have three or four significant digits.

## Scientific Notation

Scientific notation can often help us when we are confronted with an ambiguous number of significant digits because every digit used in “normalized” scientific notation is significant. (“Normalized” means one non-zero digit on the left of the decimal point and all the rest of the digits on the right.) The number 1230 might have three or four significant digits but when expressed as

$$1.230 \times 10^3 \ ,$$

it has four and when expressed as

$$1.23 \times 10^3 \ ,$$

it has three.

### Uncertainty

Again assuming all digits in a number a relevant and accurate, an important relationship exists between the number of significant digits and its uncertainty. If we consider only the most basic kind of uncertainty, we can say we don't know anything about the value of the digit following the last significant digit. For example,

$$25.6 \pm 0.05$$

explicitly states our uncertainty, ±0.05, about the number 25.6 and its three significant digits. The number 26.5 can range anywhere between 25.55 and 25.65.

The digit value of the uncertainty is often ±5 and it is one decimal place smaller than the last significant of the number whose uncertainty it describes. The value of writing a number with explicit uncertainty is that it allows us to devise rules on how uncertainty (and significant digits) are propagated through mathematical operations like multiplication, division, and various other mathematical functions like, sine, cosine, logarithms and so on.

### Rules of Propagation

In a calculation involving multiplication, division, trigonometric functions, and so on, the number of significant digits in the result should equal the least number of significant digits in any one of the arguments. Here are some examples.

$$253.49 \times 0.000091 = 0.23$$
$$\frac{78.881}{0.423} = 186.$$
$$\sin(22.25^{\circ}) = 0.3786$$
$$e^{1.4423} = 4.2304$$

When we add or subtract, the number of *decimal places* (not significant digits) in the answer should be the same as the least number of decimal places in any of the numbers being added or subtracted.

$$22.8 + 0.0014922 = 22.8$$
$$15.19 + 0.2199 = 15.41$$

The second example above indicates that we should perform the addition or subtraction using a few more digits than we expect in the final result. “Insignificant” digits can accumulate and become significant.

When used in a calculation, we assume that integers or constants like $e$ and $\pi$ have an “infinite” number of significant digits. So, in the formula for the volume of a sphere of radius 7.23 cm,

$$V = \frac{4}{3} \pi r^3 = 1.58 \times 10^3 \, cm^3$$

even though 4 and 3 seem to have only one significant digit, we treat them as if they have many. Then the number of significant digits in the result is controlled only by the number of significant digits in the radius. Also, we should use a few extra digits to perform multiplication/division and then round the result to the correct number of digits.

## *Developing Rules of Propagation*

When we attempt to develop the rules for propagation of uncertainty, we will discover we need a formula that tells us the number of significant digits in a number. Suppose we write a number with its uncertainty in a very general way like

$$x \pm \delta_x$$

where $x$ is the number and $\delta_x$ is its uncertainty. We know that the logarithm of a number greater than one counts the powers of 10. That counting corresponds to the number of digits on the left side of the decimal point and is almost what we want. We can scale the number by dividing it by its uncertainty and then we should have a formula that predicts the number of significant digits $n_x$. That is,

$$n_x = \log\left(\frac{x}{\delta_x}\right)$$

The table below shows a few examples.

| $x$ | $\log(x/\delta_x)$ | **Sig. Digits** |
|---|---|---|
| 26.1±0.05 | 2.718 | 3 |
| 0.593±0.0005 | 3.074 | 3 |
| 7184.±0.5 | 4.157 | 4 |
| 9990±5 | 3.3009 | 3 |
| 10000.0±0.05 | 5.301 | 6 |
| 123432.00±0.005 | 7.392 | 8 |

The formula is not very robust because when x is negative, the logarithm is undefined. When x is near zero the formula predicts one significant digit. We can improve the formula by using the absolute value of x in the numerator. Repairing the “near zero” problem is harder and introduces complexities that are not helpful in our discussion, so we will just assume that we never use arguments near zero. The reader may wish to revisit this discussion later and attempt to justify this omission.

## *Significant Digits in Addition/Subtraction*

Suppose we have two numbers, $a$ and $b$, that we wish to add. Each number has an associated uncertainty, $\delta_a$ and $\delta_b$. When we add, we find

$$(a \pm \delta_a) + (b \pm \delta_b) = (a + b) \pm (\delta_a + \delta_b) \ .$$

That is, the result is the sum of $a$ and $b$ but it has an uncertainty that is is sum of the uncertainties of $a$ and $b$. (Note that we have assumed that the values for $\delta_a$ and $\delta_b$ are positive.) The uncertainty in the sum is the sum of the uncertainties. Expressed mathematically,

$$(a \pm \delta_a) + (b \pm \delta_b) = c \pm \delta_c$$

where

$$c = a + b$$
$$\delta_c = \delta_a + \delta_b$$

We can now state a rule of thumb for addition and subtraction. “The result of an addition/subtraction has an uncertainty at least as large as the uncertainty of the most uncertain of the arguments”. This statement is the equivalent of the addition statement in the “Rules of Propagation” because, in this case, specifying the uncertainty is the same as specifying the number of decimal places.

## *Significant Digits in Multiplication/Division*

We can consider multiplication and division using a similar notation. Since division can always be expressed as a multiplication, we will consider only multiplication. Suppose we have two numbers, $a$ and $b$, that we wish to multiply. Each number has an associated uncertainty, $\delta_a$ and $\delta_b$. When we multiply, we find

$$(a \pm \delta_a)(b \pm \delta_b) = ab \pm b\delta_a \pm a\delta_b \pm \delta_a \delta_n \ .$$

We can rewrite this expression as

$$(a \pm \delta_a) \cdot (b \pm \delta_b) = c \pm \delta_c$$

where

$$c = ab$$
$$\delta_c = \pm b\delta_a \pm a\delta_b + \delta_a \delta_b$$

We will specify the number of significant digits in the result using the expression we developed earlier. That is,

$$n_c = \log\left(\frac{c}{\delta_c}\right) \ .$$

To make the mathematics a little easier, we should consider the inverse ratio, $\delta_c/c$ with the caveat that $a$ and $b$ cannot be zero.

$$\frac{\delta_c}{c} = \frac{\pm b\delta_a \pm a\delta_b + \delta_a \delta_b}{ab} = \pm\frac{\delta_a}{a} \pm \frac{\delta_b}{b} + \frac{\delta_a \delta_b}{ab}$$

Considering the three terms on the far right, notice that they all have factors of $\delta_x/x$ where $x$ can be $a$ or $b$ and therefore each of the factors is referencing only one of the multiplicands. Each of these factors is less than one and it quickly shrinks as more significant digits are used in the multiplicand it references. The last term will always be much smaller than the other two because the product of two small numbers is a very small number. We can safely ignore the last term in comparison with the other two. We also need to be cautious with the signs. We cannot find the logarithm of a negative number so we will only use the plus signs in the expression. This convention should not change our final result. With these observations, we can write

$$n_c = \log\left(\frac{c}{\delta_c}\right) = -\log\left(\frac{\delta_c}{c}\right) \approx -\log\left(\frac{\delta_a}{a} + \frac{\delta_b}{b}\right)$$

If the two multiplicands have the same number of significant digits, then $\delta_a/a$ and $\delta_b/b$ will be about the same size. We can then write

$$n_c \approx -\log\left(2\frac{\delta_a}{a}\right) = -\log\left(\frac{\delta_a}{a}\right) - \log(2) = -\log\left(\frac{\delta_a}{a}\right) - 0.30$$

Notice that the expression -log($\delta_a/a$) is the number of significant digits in first multiplicand (and also the second). The -0.30 term can be safely ignored because it represents about 1/3 of a significant digit. The result says that the number of significant digits in the answer is the same as the number of significant digits in the arguments.

If one of the numbers, say $b$, has more significant digits than $a$, $\delta_a/a$ will generally be much larger than $\delta_b/b$. In this case, the $\delta_b/b$ term can be safely ignored. Assume the larger term is $\delta_a/a$. Then we get

$$n_c \approx -\log\left(\frac{\delta_a}{a}\right)$$

This result says that if $a$ has fewer significant digits than $b$, the result has the same number of significant digits as $a$.

If $a$ has more significant digits, we can exchange the $a$ and $b$ labels and get the same equation above but in each case, $a$ represents the number with the fewer significant digits.

The rule of thumb for multiplication/division can now be stated as follows: “The result of a multiplication will have the same number of significant digits as has the argument with the fewer number of significant digits”. This statement is equivalent to the multiplication rule stated in the “Rules of Propagation” section.

## *Significant Digits in Functions*

## Sine and Cosine

Let's begin with a simple equation involving the sine function.

$$y = \sin(\theta)$$

Assuming θ and y will have some uncertainty, we can write a more general expression including the uncertainties.

$$y \pm \delta_y = \sin(\theta \pm \delta_\theta)$$

Expanding this equation, we have

$$y \pm \delta_y = \sin(\theta)\cos(\delta_\theta) \pm \cos(\theta)\sin(\delta_\theta)$$

We will assume that $\delta_\theta$ is always close to zero so cos($\delta_\theta$) will always be close to 1. Similarly, since $\delta_\theta$ is always close to zero know that sin($\delta_\theta$) can be replaced by $\delta_\theta$ (when is $\theta$ expressed in radians). Then we can rewrite the previous equation as

$$y \pm \delta_y = \sin(\theta) \pm \cos(\theta)\delta_\theta$$

We can then make the following associations from the preceding equations,

$$y = \sin(\theta)$$
$$\delta_y = \cos(\theta)\delta_\theta$$

Since the number of significant digits in y can be expressed as

$$n_y = \log\left(\frac{y}{\delta_y}\right)$$

$$n_y = \log\left(\frac{\sin(\theta)}{\delta_\theta \cos(\theta)}\right) = \log\left(\frac{\tan(\theta)}{\delta_\theta}\right)$$

Because log(0) is not defined and because tan($\theta$) grows large at π/2 we will restrict the values of $\theta$ to

$$0 < \theta < \pi / 2$$

However, we can extend our final results to almost any value of $\theta$ using trigonometric relations.

For values of $\theta$ near zero we can approximate tan($\theta$) with $\theta$ and then we may write

$$n_y = \log\left(\frac{\tan(\theta)}{\delta_\theta}\right) = \log\left(\frac{\theta}{\delta_\theta}\right) = n_\theta$$

The rule of thumb is then, "The number of significant digits in the result of the sine function is the same as the number of significant digits in its argument". But the equation also indicates that the number of significant digits near π/2 can be large. I believe this is true because the value of the sine function near π/2 is very nearly 1 and it is changing very slowly with $\theta$. So even if the uncertainty in $\theta$ is large, the uncertainty in sin($\theta$) grows smaller near π/2. We could, in fact, use more significant digits in that region. However, doing so would make our rule of thumb complex and unwieldy. So let's leave the current rule of thumb as it is and be happy.

The arguments for the cosine function proceed in an identical manner so they will not be presented. Since

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \ ,$$

we could apply the arguments for the sine and cosine function and then apply the argument for multiplication and, again, get the same result. That is, the result from a trigonometric function will have the same number of significant digits as its argument.

## Exponential

The exponential function is a special case of exponentiation. So the arguments in this section are valid whenever we use any form of exponentiation. We denote the exponential function as

$$y = e^x \ .$$

If we think of uncertainty in the argument $x$ as $\delta_x$ we can write

$$e^{x+\delta_x} = e^x \cdot e^{\delta_x}$$

In all our previous analyses, the uncertainty was additive. In this case, it is multiplicative. We can convert it to an additive form with an advance technique. If we assume that $\delta_x$ is very nearly zero, we can use a few terms in a Taylor series expansion of the exponential function to approximate the value of the error coefficient. (A Taylor series expansion is a technique that requires calculus to fully understand but we can show the result.) That is, the Taylor series expansion of the exponential function about zero is

$$e^{\delta_x} = \frac{(\delta_x - 0)^0}{0!} + \frac{(\delta_x - 0)^1}{1!} + \frac{(\delta_x - 0)^2}{2!} + \frac{(\delta_x - 0)^3}{3!} + \ldots$$

$$e^{\delta_x} = 1 + \delta_x + \frac{\delta_x^2}{2} + \frac{\delta_x^3}{6} + \ldots$$

Powers of $\delta_x$ are extremely small when compared to 1 and $\delta_x$ so we will ignore them and consider only the first two terms of this expansion. That gives us

$$e^{x+\delta_x} = e^x \cdot (1 + \delta_x) = e^x + \delta_x e^x$$

and we can identify

$$y = e^x$$

$$\delta_y = \delta_x e^x$$

The second equation says the uncertainty in result of the exponential function is the uncertainty of the argument but increased in proportion to the result. That is, the number of significant digits in the result is the same as the number of significant digits in the answer. A quick example will show how the error propagation works.

$$x = 7.43 \pm 0.005$$

$$e^x = 1685.81 \pm \delta_y$$

$$\delta_y = 0.005 \cdot 1685.81 = 8.429$$

$$e^x = 1686 \pm 8$$

In other words, the argument, 8.43±0.005, has 3 significant digits and the result, 1686±8, also has three significant digits.

## *Summary*

Each function we examined, multiplication, trigonometric function, and exponential, has shown the same result: "The number of significant digits in the result is the same as the number of significant digits in its argument". However, the details about how we showed this result varied with each function and there is no guarantee that this results holds for all functions. If fact, it is a trivial exercise to show exceptions to the result.

When we encounter a new function, our first assumption should be that the function follows the general result: "The number of significant digits in the result of a function is the same as the number of significant digits in its argument".