COORDINATES, TIME, AND THE SKY

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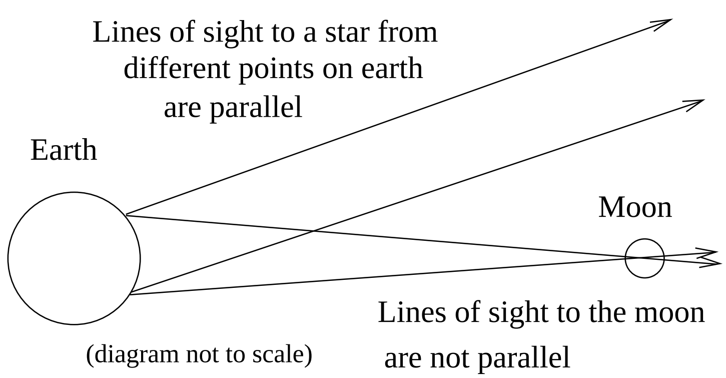
This subject is fundamental to anyone who looks at the heavens; it is aesthetically and mathematically beautiful, and rich in history. Yet I'm not aware of any text which treats time and the sky at a level appropriate for the audience I meet in the more technical introductory astronomy course. The treatments I've seen either tend to be very lengthy and quite technical, as in the classic texts on 'spherical astronomy', or overly simplified. The aim of this brief monograph is to explain these topics in a manner which takes advantage of the mathematics accessible to a college freshman with a good background in science and math. This math, with a few well-chosen extensions, makes it possible to discuss these topics with a good degree of precision and rigor. Students at this level who study this text carefully, work examples, and think about the issues involved can expect to master the subject at a useful level. While the mathematics used here are not particularly advanced, I caution that the geometry is not always trivial to visualize, and the definitions do require some careful thought even for more advanced students.

Coordinate Systems for Direction

Think for the moment of the problem of describing the direction of a star in the sky. Any star is so far away that, no matter where on earth you view it from, it appears to be in almost exactly the same direction. This is not necessarily the case for an object in the solar system; the moon, for instance, is only 60 earth radii away, so its direction can vary by more than a degree as seen from different points on earth.



But for stars and more distant objects we can ignore this complication – for a first approximation we need only specify the direction of the star, rather than its full three-dimensional position in space.

To specify a direction in space, we use *celestial coordinates*. These are broadly analogous to the familiar Cartesian 'x-y' coordinates you know about – one specifies some numbers, and these serve to specify what you want, in this case a direction in space rather than a point on a plane. A key idea in what follows is that *directions in space map in a straightforward way onto points on a sphere.* To see this, imagine drawing vectors from the center of a sphere to its surface; vectors drawn in different directions will intercept the sphere at different points, and different points all lie in different directions from the center – in other words, there is a one-to-one correspondence between directions and points at the surface of the sphere. (That's true of any convex figure, but a sphere is especially convenient.) Because of this correspondence, it's conventional to imagine an *arbitrarily large* sphere around the earth to represent the *directions* of objects in space. This sphere is called the *celestial sphere.* It is purely a mathematical construction, with no physical reality, but it is an extremely powerful conceptual tool.

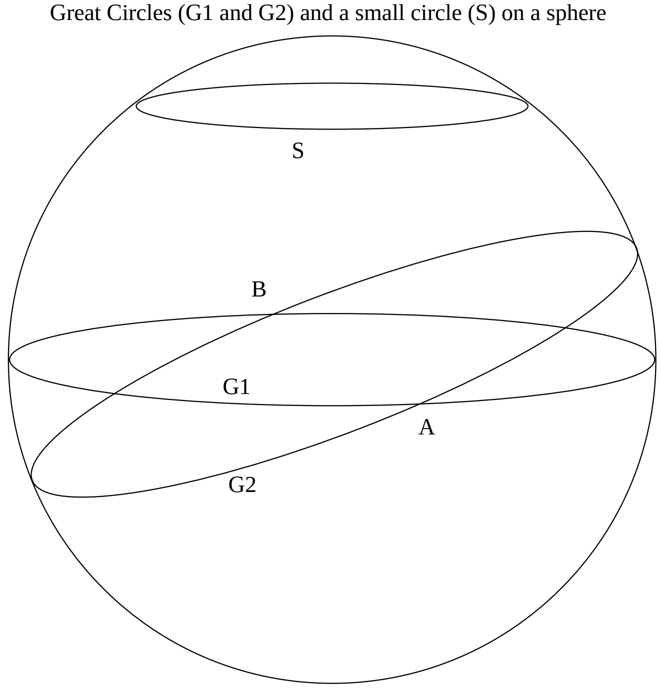
So, our problem of representing directions in space reduces nicely to the problem of coming up with coordinates to represent points on the surface of a sphere. This problem is already familiar from geography, which uses latitude and longitude. We apply a broadly similar set of coordinates to the sky.

Before discussing coordinates, we should explore some aspects of spherical geometry, which is not usually covered well in standard mathematics courses. First, let's think a little bit about angles in general. In a circle drawn on a flat piece of paper, it's obvious that an *angle* measured at the center of a circle is proportional to an *arc* measured along the circle. If θ is the angle in radians, R is the radius of the circle, and A is the length of the arc, then

$$A = \theta R.$$

This is also true of arcs measured along the surface of a sphere, but in spheres there is an important distinction between a *great circle* and a *small circle*. A *great circle*

Great Circles (G1 and G2) and a small circle (S) on a sphere



is a circle drawn on the surface of a sphere, the plane of which passes through the center of the sphere; by contrast, the plane containing a *small circle* does not pass through the exact center of the sphere (close doesn't count!). An an example, every line of constant longitude on the earth is a part of a great circle, as is the equator; but every line of constant latitude *except* the equator is technically a small circle, even though some of them are nearly as big as the equator itself.

Here are some properties of great circles which are useful and help illustrate what a great circle is.

- A great circle divides the surface of a sphere into two exactly equal parts.
- If any two great circles intercept, they intercept at two points which are exactly opposite each other (like points A and B in the diagram).
- The shortest distance on the surface of a sphere between any two points on the surface is along the great circle which connects the two points.

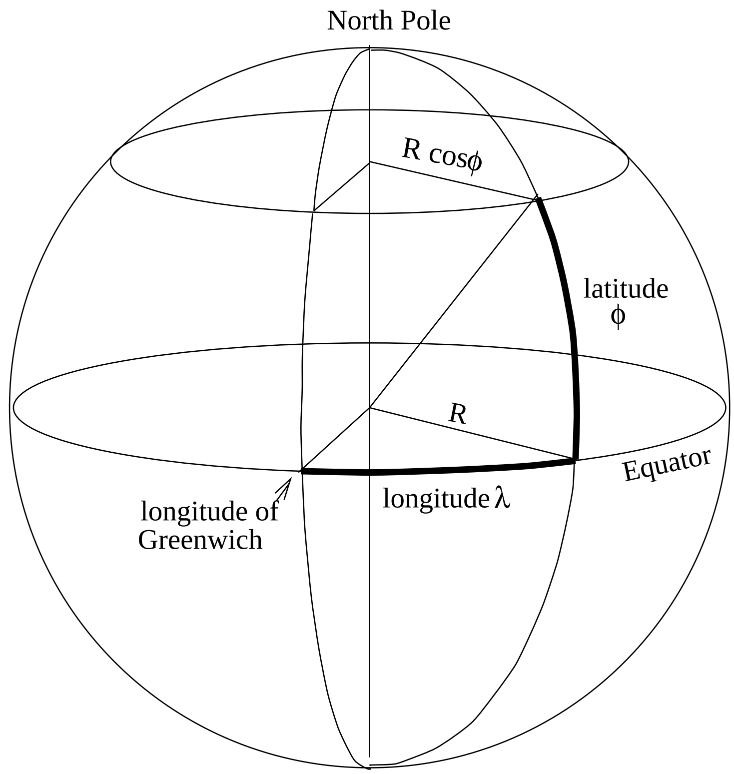
Great circles are also useful because of their connection with angles. We'll often be interested in the angular distance between objects – the angle between the directions to the two objects. Because we live (by definition) at the exact center of the celestial sphere, the angular distance between two objects is the angle subtended by the two objects at the center of the celestial sphere – in other words, the angle between lines drawn from the objects to our position at the center of the sphere. If we draw a great circle which passes through the two objects, the length of that arc will be $A = \theta R$, where R is the arbitrary radius of the celestial sphere. Therefore the arc length along the great circle connecting the objects' positions is directly proportional to the angular distance between the two objects. Because the radius R of the celestial sphere is arbitrary, we can effectively ignore it by calling it unity (one), and treat the arc length as being the same thing as the angle. So we may use a great-circle arc lengths as a proxy (or 'stand-in') for the angle subtended by two objects. This makes great-circle arcs especially useful.

Arcs along any small circle still subtend an angle at the center of the circle, and that angle is (again) A divided by the radius, but now the appropriate radius is that *of the small circle*, which is no longer the radius of the sphere. And the center of the sphere is no longer the center of the circle along which we're measuring, so angles along a small circle do not correspond directly to angles subtended between objects.

Now we can look a little more closely at the coordinate systems we use for spheres. The usual way of specifying points on a sphere's surface is through a *spherical-polar coordinate system*. Latitude and longitude are the most familiar example, so we'll use them as our first example. A spherical-polar system depends on having a *pole*, such as the earth's north pole. This in turn defines an *equator*, which is the set of all points 90 degrees away from the pole. One then selects a *zero point* somewhere along the equator; for geographical coordinates, the zero point is the longitude of the original Royal Greenwich Observatory near London. There's no compelling scientific reason for this – it's just an accident of the ebb and flow of imperial power.

The two coordinates used on earth are *latitude* (denoted ϕ), which is angular distance from the equator (positive north, negative south), and *longitude* (denoted λ), which is angular distance along the equator from the zero point to the east-west position of the point, as shown in the accompanying figure. Note that the equator is the only east-west great circle, and the great-circle arc makes arcs and angles equivalent. A relatively recent convention is that longitude λ is to be measured positive eastward, and negative westward. Notice how lines of constant longitude grow closer together towards the pole. The radius of the small circle at latitude ϕ is $R\cos\phi$, where R is the radius of the earth. So the length of a small-circle arc between longitudes λ_1 and λ_2 at latitude ϕ is

$$\text{arc length at latitude } \phi = R(\lambda_2 - \lambda_1)\cos(\phi).$$



Equatorial Coordinates - RA and dec

Now it's time to finally introduce the most important set of celestial coordinates, which are called *equatorial coordinates*. As I remarked above, spherical-polar coordinates are angles, and they require that you specify a *pole*. The pole used for equatorial coordinates is the direction of the *earth's axis*. The point where the direction of the earth's axis – the north part, that is – intercepts the celestial sphere is called the *North Celestial Pole*, which we'll abbreviate *NCP*. In a time exposure of stars near the NCP taken with a camera fixed to the ground, the stars make arcs centered on the NCP. Another picturesque way of thinking of the NCP is to imagine placing an infinitely long stick through the earth along its axis; if you view this stick from someplace in the earth's northern hemisphere, it will appear to rise above the horizon in the north and extend off into the distance. By the laws of perspective its far end will disappear at the NCP – the direction of the earth's axis. The NCP lies very close to the star Polaris (which, contrary to many people's impressions, is not particularly conspicuous). Southern hemisphere observers see a *South Celestial Pole*, which doesn't have any bright star near it.

If you stand on the north pole of the earth, the north *celestial* pole is directly overhead in the sky – it lies in your *zenith*, which is another name for the point straight up in the sky. If you stand on the equator, the north celestial pole lies on the horizon, due north, and the south celestial pole lies on the horizon due south. At any intermediate latitude, the (smallest) angle between your horizon and the north celestial pole is just equal to your geographic latitude. So at Kitt Peak, in Arizona, which is at a geographic latitude of 32 degrees, the north celestial pole is 32 degrees above the horizon. The angular distance of an object above the horizon is called its *altitude*; the altitude of the celestial pole is equal to your latitude. Note that altitude is measured along a great-circle arc which passes through the object and the zenith.

The existence of a pole implies the existence of a *celestial equator*, which is the set of all directions 90 degrees from (either) pole. If you stand on the north or south poles, the celestial equator is identical to your horizon. If you stand on the earth's equator, the celestial equator is perpendicular to your horizon and intercepts the horizon at the due east and due west points, and passes through your zenith. At intermediate latitudes, the celestial equator still crosses the horizon due east and due west, but the angle it makes with the horizon is equal to your geographic *colatitude*, which is just 90 degrees minus your latitude. At its highest point the celestial equator's altitude is equal to your colatitude.

Now that we have our pole and equator defined, we can introduce our two coordinate angles, *right ascension* and *declination*. Right ascension is sometimes abbreviated RA, and it is standard to use the Greek letter α for right ascension. Declination is sometimes abbreviated as dec, and the Greek letter δ is used.

Right ascension is the *longitude-like* coordinate – it measures east-west position. Declination is the *latitude-like* coordinate – it measures north-south position. Just as there is an arbitrary zero point for longitude on earth (the observatory at Greenwich, near London) there is a zero point for right ascension. This is called the *First Point of Aries*. As it turns out, it is not arbitrary, but we may consider it so for now. The zero point of declination is not arbitrary, but is rather the celestial equator.

Right ascension increases *eastward* on the sky. Declination increases *northward* on the sky; northern declinations are positive and southern declinations are negative.

RA and dec can be specified in any units used for angles – they can be degrees, radians, or whatever. However, there is a custom which is still used today, and while this seems arcane at first there are some good arguments for keeping it.

- Declination is measured in *degrees, minutes, and seconds*. The minutes and seconds used for degrees are called *minutes of arc* and *seconds of arc*, and are often denoted *arcmin* and *arcsec*. The relation is the same as with time units – an arcmin is 1/60 of a degree, and an arcsec is 1/60 of an arcmin, so an arcsec is 1/3600 of a degree. The notation used for degrees, minutes, and seconds is $^\circ$ for degrees, $'$ for arcmin, and $''$ for arcsec, so that 31 degrees, 57 arcmin, and 12.3 arcsec would be written as $31^\circ\ 57'\ 12''.3$. Note that the $''$ is written by the decimal point.

- Right ascension is measured in a different unit, in which a full circle is 24 units. In other words, the full circle is divided just as the day is divided into hours, and the *angular* unit for right ascension is named identically to the *time* unit – it's called the *hour*. An hour of right ascension is divided into minutes and seconds just like the degree (and like the usual hour). Although these minutes and seconds are really being used to measure angles, they are called *minutes and seconds of time* to differentiate them from portions of a degree. Hours, minutes and seconds are denoted as $^{\mathrm{h}}$, $^{\mathrm{m}}$, and $^{\mathrm{s}}$, so that a right ascension of 4 hours, 42 min, 32.33 sec would be written as $4^{\mathrm{h}}\ 42^{\mathrm{m}}\ 32^{\mathrm{s}}.33$. Note that the superscript $^{\mathrm{s}}$ is written together with the decimal point. Right ascensions run from 0^{h} to 24^{h}, where they wrap around back to zero.

  A little thought shows that if 1 circle = 360 degrees = 24 hours, then

$$1 \text{ hour} = 15 \text{ degrees.}$$

And because the structure of degrees and hours is exactly parallel,

$$1 \text{ minute of time} = 15 \text{ arcmin}$$

and

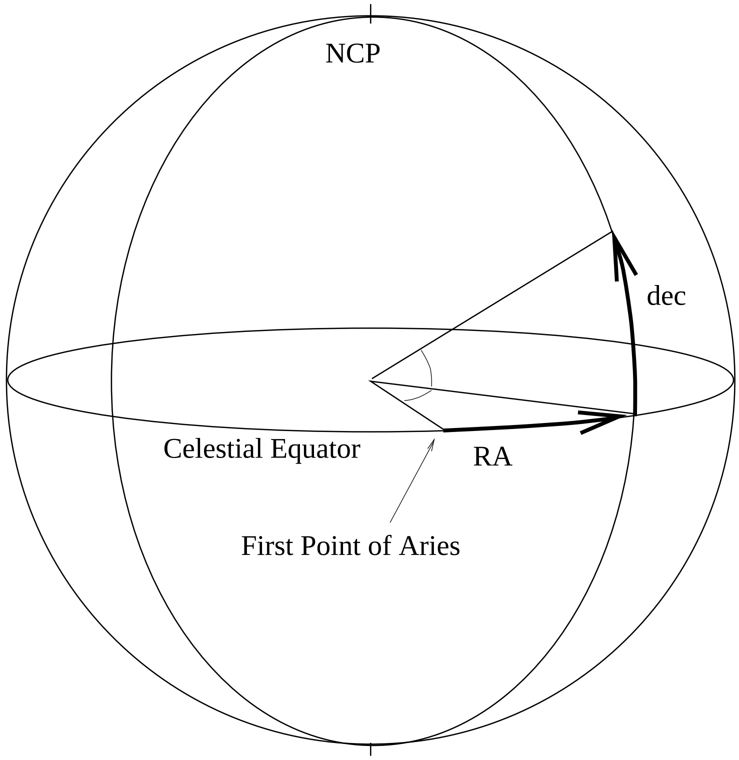
$$1 \text{ second of time} = 15 \text{ arcsec.}$$

The structure of 60s used for time and angles is called *sexigesimal* notation. It is a little tricky because one must convert a sexigesimal triplet into an equivalent decimal number before doing any arithmetic. Luckily, most 'scientific' calculators have some kind of 'hours to decimal' conversion built in*. So the first step in doing any problem which involves taking (say) trig functions of a right ascension is to convert the sexigesimal right ascension to decimal hours, and then convert to radians or degrees (as appropriate) before evaluating the trig function.

You will recall that I made a big deal about how *angles* are equivalent to *arcs* along great circles. The figure shows how right ascension and declination are measured as arcs – right ascension is measured along the equator, eastward, and declination is northward. The angles subtended at the center of the sphere – where we observe – are shown drawn

* Computer programs can be arranged to do these conversions at input and output, but they should note that, in a small band just south of the equator, the leading field in a declination will be '−0' which evaluates to a positive number; the minus sign must be converted separately as a character on input.

in lightly.



The system of right ascension and declination is nearly fixed in space – it's a first approximation to what physicists call an *inertial reference frame*, which is not accelerating or (more appropriately for a system which specifies only *directions*) it is not *rotating*. As we'll see later, it isn't quite inertial, because the direction of the earth's axis which defines the system is not perfectly constant on long time scales.

The meridian, hour angle, and sidereal time

Now we turn our attention more closely to what an observer on the earth sees.

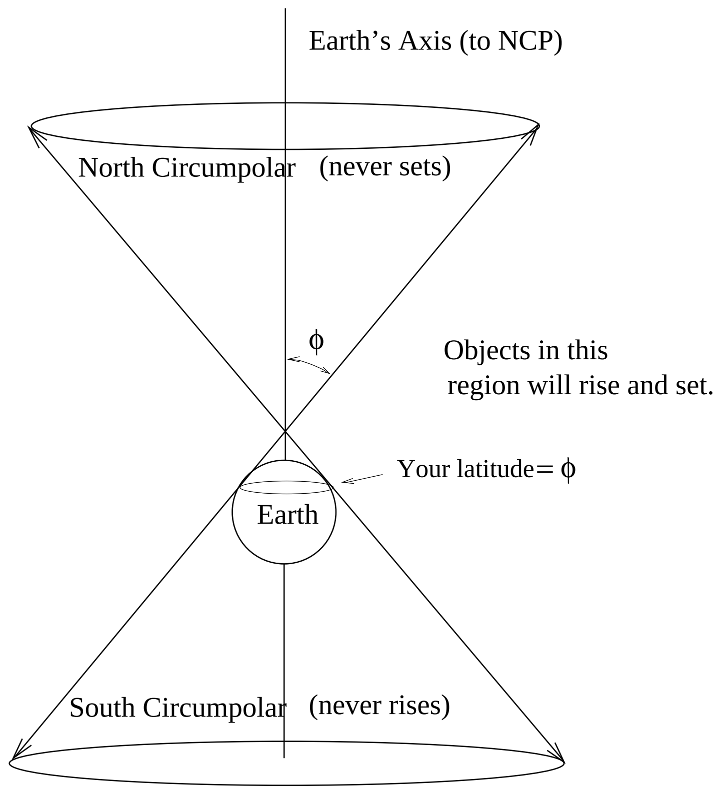
From a level site (such as the ocean) you can see half the celestial sphere at any one time. The horizon is a great circle, so it divides the celestial sphere into two parts.

One can imagine the horizon as a plane which is tangent to the earth at the point you are standing. The circle at which this plane intercepts the celestial sphere divides the celestial sphere into visible and invisible portions. Because the celestial sphere is infinitely large, the size of the earth doesn't matter, so the geometrical horizon splits the celestial sphere into two precisely equal parts *.

As the earth rotates, your horizon plane rotates with it. The diagram shows how the plane rotates in space. For a mid-northern observer, some directions toward the north (for a northern observer) will always be to one side of the plane through the entire rotation – these

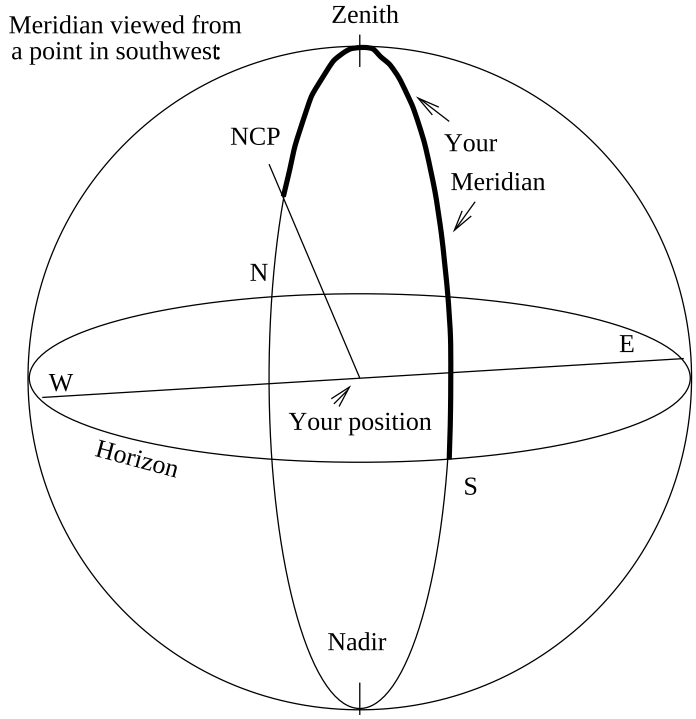
* The actual horizon is affected slightly by atmospheric refraction

directions are called *north circumpolar* directions. The extent of the north circumpolar region depends on your latitude; north circumpolar directions lie inside a cone whose axis is parallel to the earth's axis and whose opening half-angle is equal to your latitude. A similar cone toward the south – the *south circumpolar* region – never rises above the horizon. All directions in between will rise and set with each rotation of the earth.



As I mentioned earlier, if one takes a time exposure of the stars around the NCP with the camera fixed, the star images make big arcs around the NCP, because of the rotation of the earth. The arcs which just graze the horizon delineate the extent of the circumpolar region.

Now imagine a great circle which splits your observed sky into *east-west* halves. This would come out of the horizon at the due south point, pass through your zenith, through the north celestial pole (for northern sites), and intercept the horizon again at the due north point. This arc is called your *meridian*. It is fixed in your sky, not fixed among the stars. As the earth turns, stars will appear to pass by your meridian. Your meridian will change if you travel east and west, but will remain the same if you travel due north or south. When we say 'meridian' we will generally mean only that part of the meridian which stretches from the celestial pole, across the zenith, and down to the southern horizon (for northern sites). The figure illustrates this definition of meridian drawn on a celestial sphere; note that the horizontal circle (drawn as an ellipse to show perspective) represents your horizon. One can do this because the celestial sphere is a sphere of arbitrary size which represents *directions*; the horizon shown is just the set of directions which correspond to your horizon.



A star on your meridian is as high as it will ever get in your sky. If the star happens to be at a declination equal to your latitude, it will pass directly through the zenith when it appears on your meridian.

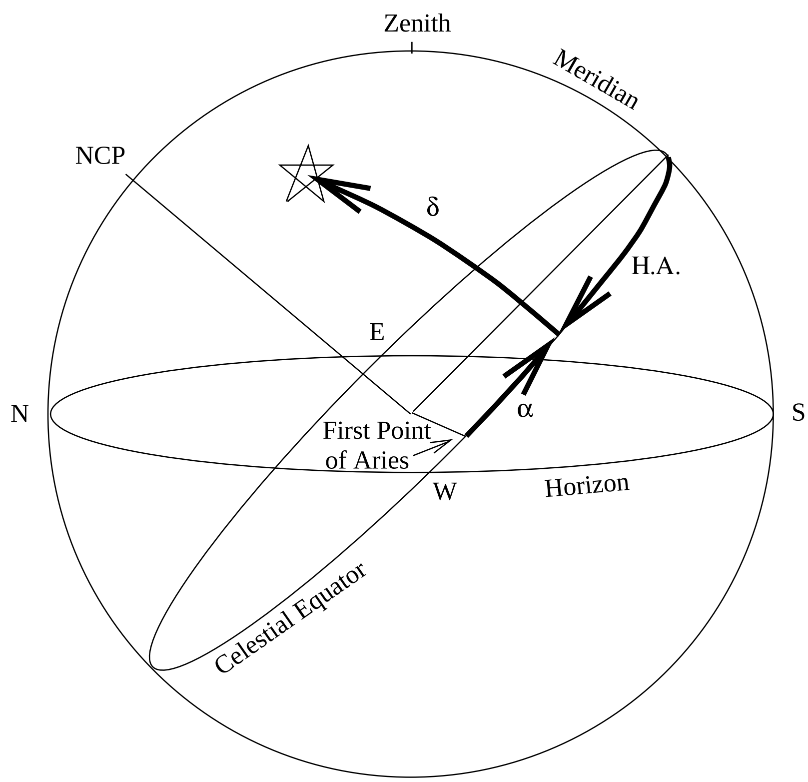
As I mentioned, as the earth turns from west to east, stars appear to pass across your meridian from east to west. The right ascension toward which your meridian points will change at a constant rate. Anything which changes at a constant rate can be used to measure time. The right ascension of the point on your meridian is therefore a sort of clock, which is used to define a new kind of time, called *Sidereal Time*. The sidereal time is simply the right ascension of your meridian. This is why right ascensions are customarily measured in hours – the constant rate of rotation of the earth makes for a natural connection between east-west angles and time. In fact, it's a shame that geographical longitudes are not measured in hours, because then it would be simpler to see the relation between different time zones on the earth's surface. We'll explore the relationship between sidereal time and regular 'solar' time a little later. In the meantime, you might note that because your meridian is a strictly local quantity (it's different in Boston than it is in Buffalo), the sidereal time is also strictly local.

But first it's time to define yet another angle. You'll recall how a spherical polar system requires a pole and an arbitrary zero point along the equator from which to measure the longitude-like coordinate. As the sky appears to slip westward, the zero point of right ascension (the first point of Aries) appears to go round and round when viewed from a point on the earth. Now, suppose that we take our meridian as the zero point of a new longitude-like coordinate, while retaining the NCP as our pole. Note that this new longitude-like coordinate and RA will continually slip past each other, exactly as stars continually cross over the meridian from east to west.

The longitude-like coordinate for this system is called *Hour Angle*. Like right ascension, it is measured along the equator, but now it is measured in the opposite sense – positive *westward*. Hour angle is customarily measured in hours, minutes, and seconds, just as is right ascension. The hour angle of an object on the meridian is zero; as it moves westward, its hour angle increases up to +12 hours, at which point it is said to be at *lower culmination*. At that point its hour angle switches to −12 hours and counts back down to zero on the meridian.

The hour angle of the sun is roughly equal to the time of day, if you take noon as zero and adjust times in the morning appropriately. For instance, at 9 AM the hour angle of the sun is about −3 hr; at 3 PM, it is about +3 hr.

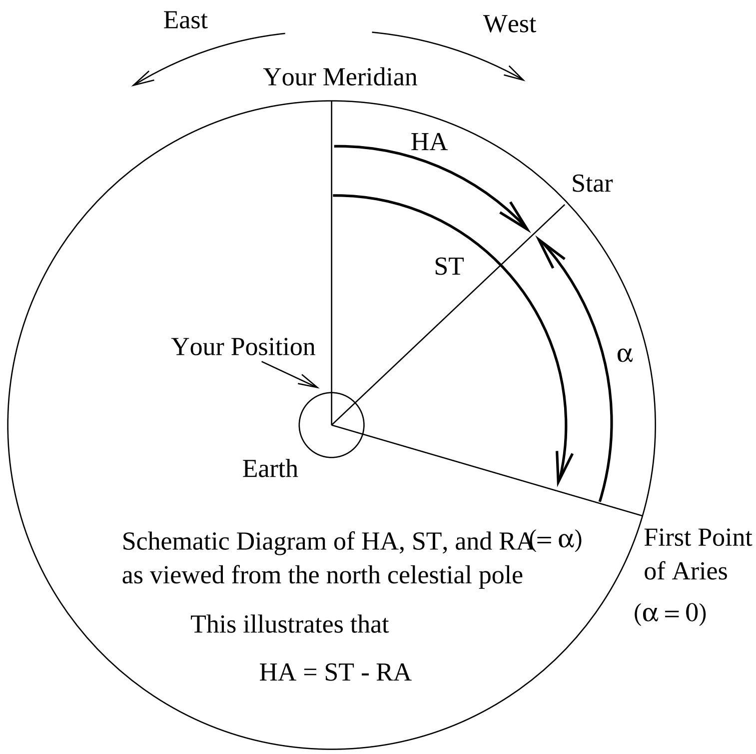
This diagram shows the hour angle, the right ascension, and the declination of a star west of the meridian and north of the equator.



Hour angle, sidereal time, and right ascension are all tied together in a single relation, which you should commit to memory if you think you'll ever use this stuff:

$$\text{Hour angle} = \text{Sidereal Time} - \text{Right Ascension}.$$

This encapsulates the whole discussion. Note that when HA = 0, ST = RA – just as defined above. One can see as well that the sidereal time is equal to the right ascension of the first point of Aries. The following diagram is a schematic view of all these east-west angles as seen from the north.



Telescope Mountings

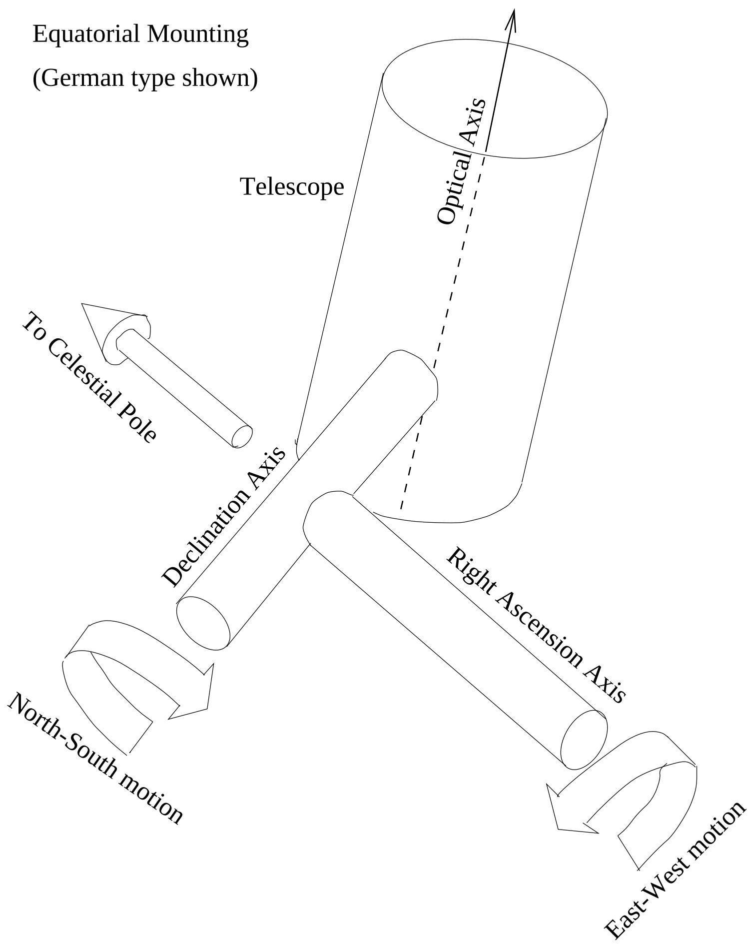
We've seen that the earth's rotation causes stars to move across the sky from east to west. A telescope mounted on an *equatorial mount* takes advantage of this symmetry. We can reinforce the concepts of celestial coordinates by considering how such a mounting is constructed.

Most of the time telescopes are mounted with two bearings perpendicular to each other. The most obvious way to orient these bearings is to have one which pivots about a vertical mounting, swinging left and right, and another, carried by the first, which swings up and down. A cannon is mounted this way, with a turret which swings left-right and an elevating mechanism which can lift the angle of the barrel. This kind of mounting is called an *altitude-azimuth, or altaz* mounting. An altaz mounting is easy to engineer, because the moving loads do not change their orientation to the bearings; the left-right axis always carries the load along the axis, and the up-down axis always carries the load perpendicular to the axis.

An altaz mounting does not track stars in a natural way. In general, a star moving across the sky moves both left-right and up-down at the same time; therefore, the telescope must be driven in both axes at once to follow a star. Furthermore, the field of view rotates. One can see that this must be the case by considering an object rising in the east; north will be to the upper left. But when the object transits later, north will be to the top.

Altaz telescopes therefore also need to be equipped with precise instrument rotators for any picture taking.

An equatorial mounting avoids these difficulties by pointing one of the axes at the north celestial pole. This makes the *polar axis* parallel with the axis of the earth. So as the earth rotates one way, the polar axis can be driven the other way at a rate of one revolution per sidereal day the other way to follow the star. The *declination axis*, perpendicular to the polar axis, carries the telescope north-south. As the mounting moves, the telescope stays in a perfectly fixed orientation *with respect to the sky*, so an instrument rotator is not needed. When showing the sky to people, I'm fond of pointing out that the telescope and its mount are the only things in the room which are *not* rotating (how could they be? They're fixed with respect to the distant stars!).



Despite the extra complication, the largest modern telescopes are on altaz mountings, because computers can now easily keep up with the calculations needed, and because the mechanical engineering of an altaz mount is so much simpler than an equatorial mounting. Amateur astronomers who intend to use their telescope only visually at low power often use altazimuth telescopes also, especially the very simple and inexpensive *Dobsonian* design pioneered by John Dobson of the San Francisco Sidewalk Astronomers. But there are a great many equatorially mounted telescopes in both amateur and professional hands.

The Apparent Motion of the Sun

From temperate and tropical latitudes the sun appears to rise in the east, cross the meridian around noon, and set in the west. This very rapid motion across the sky is of course caused by the earth's rotation on its axis; this is called the *diurnal rotation* of the earth, and the apparent motions of any bodies caused by diurnal rotation are called *diurnal motions*. They aren't real motions, of course, only apparent.

As well as rotating on its axis, the earth revolves around the sun, completing one revolution each year. The earth's orbit is an ellipse. A line drawn from the sun to the earth sweeps out a surface which lies in a plane, which is the plane of the earth's orbit. This plane is called the *ecliptic plane*. Its interception with the celestial sphere defines a great circle called the *ecliptic*.

The line of sight from the earth to the sun lies in the plane of the earth's orbit, so it lies in the ecliptic plane, and the direction of that line of sight lies on the ecliptic. So, as the earth orbits the sun, the sun appears to move along the ecliptic in the sky. If we could see stars during the daytime, we would see the sun gradually changing its position against the background of the much more distant stars, because we view the sun from a moving platform. Even though we cannot see stars during the daytime, it is easy to imagine a procedure by which we could infer the position of the sun as referred to the distant stars. For instance, we could map the stars along the ecliptic by measuring their right ascensions and declinations; then we could infer the position of the sun against the stars by (say) finding which stars are passing near the meridian at midnight. The sun's right ascension would be the right ascension of those stars minus 12 hours (half a circle).

The earth completes a full trip around the sun every year. There are 360 degrees in a circle, and a little over 365 days in a year, so the earth moves on average a little less than one degree per day along its orbit; a radius vector from the sun to the earth moves a little less than one degree per day. Therefore the line of sight from the earth to the sun — which defines the sun's position in the sky – also moves a little less than one degree per day. So the sun moves a little less than one degree per day along the ecliptic.

Just as the earth rotates from west to east, its revolution around the sun is also from west to east (that is, counterclockwise when viewed from the north). With some thought one can see that the sun must therefore appear to move from west to east along the ecliptic. So, even while the sun's diurnal motion carries it very quickly across the sky from east to west, its annual motion carries it gradually backwards against the background of the distant stars which define the non-rotating reference frame. Another way of looking at this is that the apparent diurnal rotation of the stars is slightly faster than the average apparent diurnal rotation of the sun. The earth's motion around the sun causes the sun to gradually lag behind the apparent motion of the stars, by about one degree per day. After one year has gone by, the sun has 'slipped' by a whole circle with respect to the distant stars. So the stars make a little over 366 diurnal cycles in the time it takes the sun to make a little over 365 diurnal cycles.

For daily timekeeping, we of course want our clocks to keep time at least approximately with the sun. If we ran our clocks too fast, the sun would appear to rise later and later each day, and pretty soon we'd be getting up in darkness; if we ran them too slow, we'd be getting up in the afternoon, and then the evening. So for the purposes of our civil timekeeping 24 hour must be exactly the average length of a solar day, or to be more precise the average time between meridian transits of the sun.

We can define this more precisely by considering a fictitious object called the *mean sun*. This is a sun which moves eastward among the fixed stars at precisely the *average* of the rate at which the real sun moves. The rate at which the real sun drifts eastward is not quite constant because of a couple of reasons we will get to eventually, but the mean sun's rate is exactly constant. Using this concept we can precisely define

$$\text{Local mean solar time} = 12 \text{ hr} + \text{HA of mean sun}.$$

Note that this is strictly local, since your meridian (which is needed to define hour angles) is strictly local.

But we've just seen that the earth's motion around the sun causes the sun to lag behind the stars. Now, sidereal time is defined as the RA of whatever is on your meridian – and the RAs of stars remain essentially fixed. Therefore, ordinary clock time – solar time – lags behind sidereal time by one full 24-hour cycle per year. A clock reading sidereal time must be constructed so that it runs slightly faster than an ordinary clock, so that it gains the requisite one cycle per year. The ratio of the rates of sidereal and solar clocks is

$$\frac{\text{sidereal rate}}{\text{solar rate}} = 1.0027379093.$$

If one converts this to a fraction of the form $(x+1.)/x$, one finds that $x = 365.2422$, which is the number of days in a year. The sidereal clock gains on the solar clock by about 3 min 56 sec per day, which accumulates to about 1/2 hour per week and 2 hours per month.

[Here's an aside about the calendar. The number of days in the year (365.2422) is not an integer – there's no physical reason for it to be an integer, since the rotation of the earth is independent of its revolution about the sun. This non-commensurability causes us to have an elaborate calendrical convention, by which the length of the year is approximated as a fraction

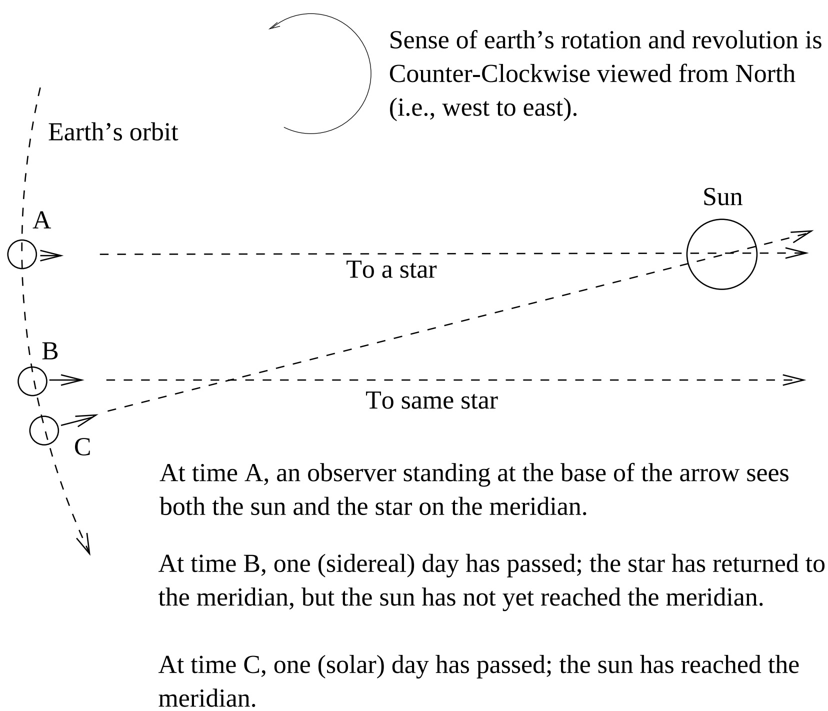
$$\text{length of calendar year} = 365 + \frac{1}{4} - \frac{1}{100} + \frac{1}{400}.$$

This is implemented as follows: An ordinary year has 365 days, but years divisible by 4 are leap years of 366 days; years divisible by 100 are an exception, and are given 365 days; but years divisible by 400 are yet another exception, and are given 366 days! So 1900 was not a leap year despite being divisible by 4, and 2000 is a leap year anyway because it is divisible by 400. With this fraction the average length of the calendar year is

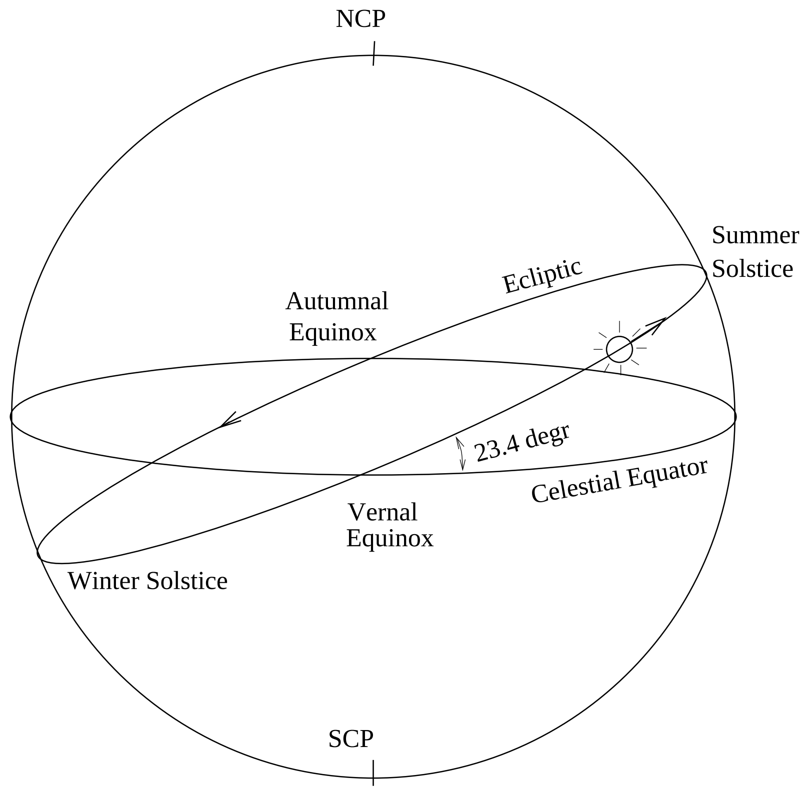
$$\text{length of calendar year} = 365.2425 \text{ day},$$

which is very close to 365.2422 day, the true length of the year.]

Back to solar and sidereal time. The diagram below presents an equivalent view as to why a sidereal clock must be made to run faster than a solar clock. The diagram depends on the fact that the distances to the stars are vastly greater than solar system distances, so that the direction of a star is essentially constant even when viewed from different points along the earth's orbit. The daily motion is highly exaggerated in the diagram for clarity; it is actually only a little less than one degree.



Thus far we've concentrated on the gradual easterly motion of the sun with respect to the fixed stars, but ignored the north-south motion. Recall that the equatorial system of coordinates is defined by the orientation of the earth's rotation axis. The ecliptic, by contrast, is defined by the orientation of the earth's orbit about the sun. These are not perfectly aligned; the earth's spin axis is tilted by about 23.4 degrees away from a line perpendicular to the ecliptic plane. This angle is called the *obliquity of the ecliptic*.



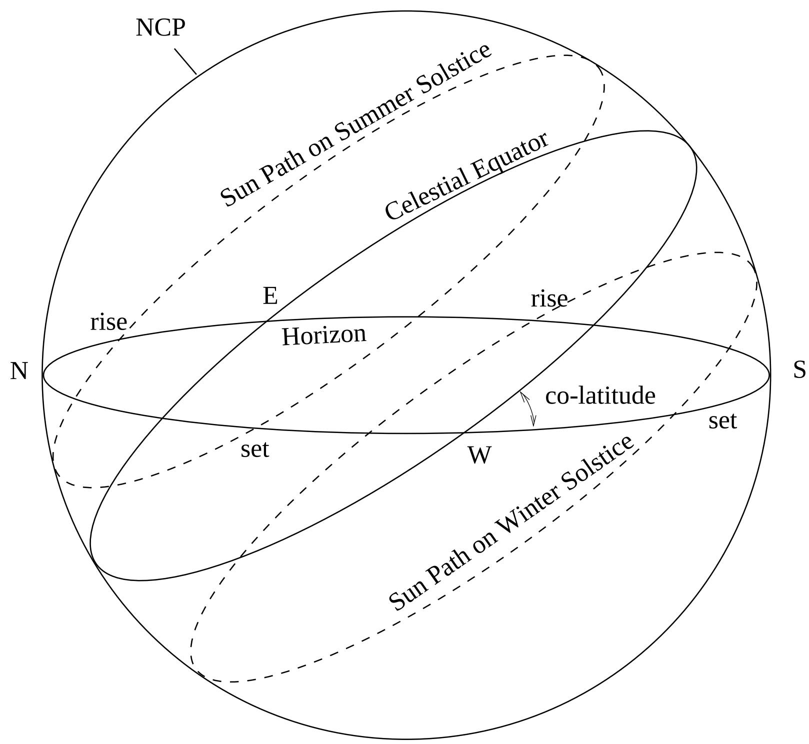
Because of the obliquity, the declination of the ecliptic ranges from zero (where it crosses the equator) to ± 23.4 degrees. Remember that the sun follows the ecliptic in its apparent annual journey around our sky. Therefore the declination of the sun varies gradually through the year within the zone $-23.4 \leq \delta \leq +23.4$.

When the sun is at its highest declination, we have the *summer solstice*; when it is at its lowest declination, we have the *winter solstice*; when it crosses the equator heading northward, we have the *vernal equinox*; when it crosses the equator heading southward, we have the *autumnal equinox*. Their approximate dates are as follows:

Vernal Equinox	March 20
Summer Solstice	June 21
Autumnal Equinox	Sept 22
Winter Solstice	Dec 21

These are UT dates (later) for 1996; there is a slight variation from year to year because the calendar doesn't line up perfectly with the year except on average.

The high declination of the sun in summer accounts for the long days and the high altitude of the sun above the horizon at noon. At the equinox, the sun is on the equator, which is a great circle. Because the horizon is another great circle, the horizon splits the daily path of the sun into two equal parts. Therefore the day and night are of equal length on the equinox – which is the origin of the name. At the winter solstice the low declination of the sun explains the shortness of the day and the low maximum altitude of the sun. So, *the obliquity of the ecliptic is the cause of the seasons.* This figure shows the apparent path of the sun across the sky for a mid-northern latitude on the summer and winter solstices – note that the horizontal plane in this diagram is the horizon, not the celestial equator. Notice how the rising point is to the northeast on the summer solstice, and to the southeast on the winter solstice; also notice that the length of the day will be proportional to the fraction of the daily path which is above the horizon, which varies dramatically with the season. Both the length of the day and the more directly vertical angle of sunlight during the summer cause the total solar energy received – the *insolation* – to be much greater during the summer than during the winter.

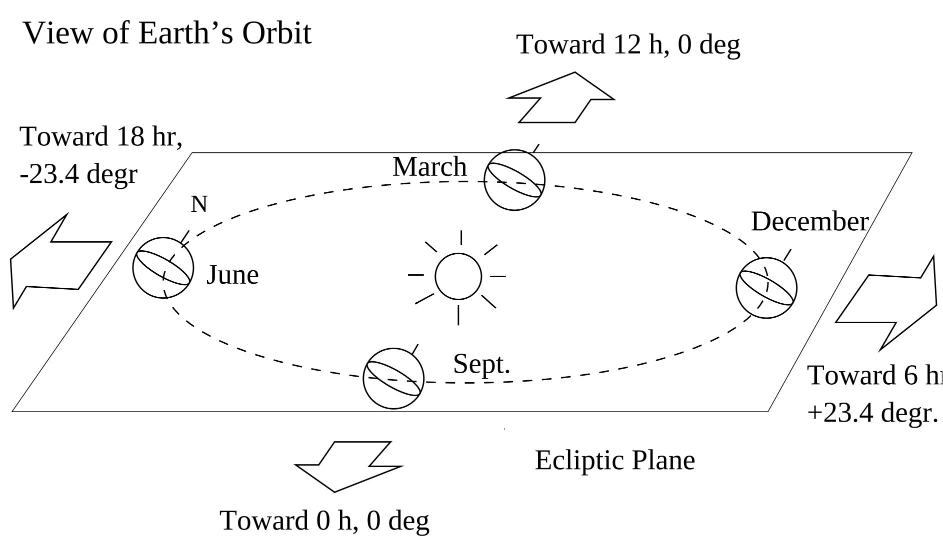


We are finally in a position to explain the zero point of the right ascension system, which until now we have left as arbitrary. The first point of Aries is at the *Vernal Equinox* – the point at which the sun crosses the equator from south to north. Therefore on the day of the vernal equinox, the sidereal time at noon is about zero hours, since the sun is at the vernal equinox, which is the zero point of right ascension. At that point the solar clock will be reading 12 hours, so we can see that at the vernal equinox the sidereal clock has gained 1/2 of a 24-hour cycle compared to the solar clock. On the opposite side of the year, the sidereal clock will read 0 hours at solar midnight – in other words, the sidereal and solar clocks agree on the autumnal equinox, and then slowly drift away from each other through the rest of the year, until they lap each other again one year later.

Astronomers who observe at night (some, such as solar and radio astronomers, do not need to) are concerned with the sidereal time at night, because that defines which parts of the sky will be observable. A rough indication of this is the *sidereal time at midnight* – objects with an RA equal to this will transit the meridian at midnight, and at that time of year one has a maximum number of nighttime hours to observe them. One can easily tell that the sidereal time at midnight will be as follows:

Vernal Equinox	March 20	12 hours
Summer Solstice	June 21	18 hours
Autumnal Equinox	Sept 22	0 hours
Winter Solstice	Dec 21	6 hours

It's now worth stepping back and looking at the whole system from afar. The next figure shows a perspective view of the earth's orbit, with the earth shown at the solstices and the equinoxes (obviously, the diagram is grossly out of scale!). Notice that in March, when one looks toward the sun, one is looking in the direction of 0 hr right ascension. At midnight one is looking diametrically away from this, toward 12 hours. It's a good idea to painstakingly correlate the table above with the diagram below! Notice also how the earth's axis maintains the same direction in space as it orbits the sun, as it must because of conservation of angular momentum; this is the underlying reason why the declination of the sun changes through the year. The little earth cartoons are tilted with respect to the ecliptic plane, which reflects the reason why the ecliptic and equatorial planes are tilted.

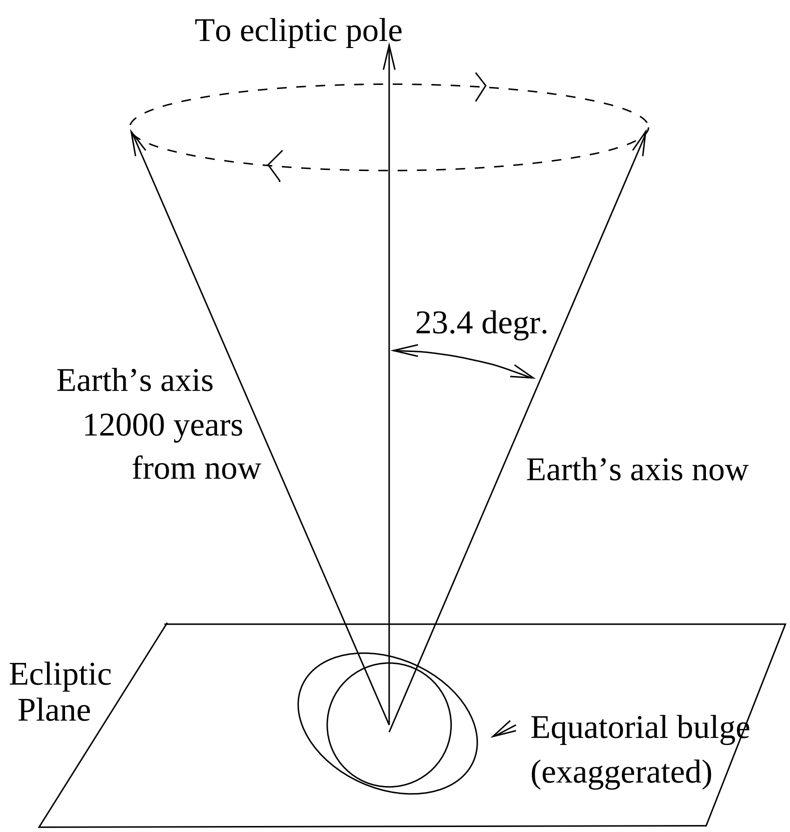


Precession of the Equinoxes

The equatorial coordinate system is tied to the axis and equator of the earth. At a gross level, the earth's axis maintains a fixed direction in space throughout the year and from year to year; as we've seen, that's the reason we have seasons.

But at a more precise level the earth's axis does not stay perfectly fixed, but rather drifts slowly in orientation. The physical reason for this is that the earth experiences a net torque from the gravitational pulls of the moon, sun, and planets. This torque arises because the earth is not quite spherical; its rotation causes it to bulge slightly toward the equator, so the earth's equatorial diameter is about 1/298 larger than its polar diameter. The equatorial bulge gives a 'handle' for the gravitational torques exerted by the sun and the moon.

These torques 'try' to twist the earth's equator back into the plane of the ecliptic. But if you've studied rotational dynamics you know that this does not lead directly to the alignment of the equator with the ecliptic, but rather to a *precession*. The earth's pole, rather than moving toward perpendicularity with the ecliptic, moves in a direction perpendicular to *both* the ecliptic pole and the earth's pole. One can see this kind of motion if you support a rapidly spinning gyroscope on a stand – rather than toppling over, it moves perpendicular to the direction in which gravity is trying to pull it, and its axis slowly describes a conical figure.



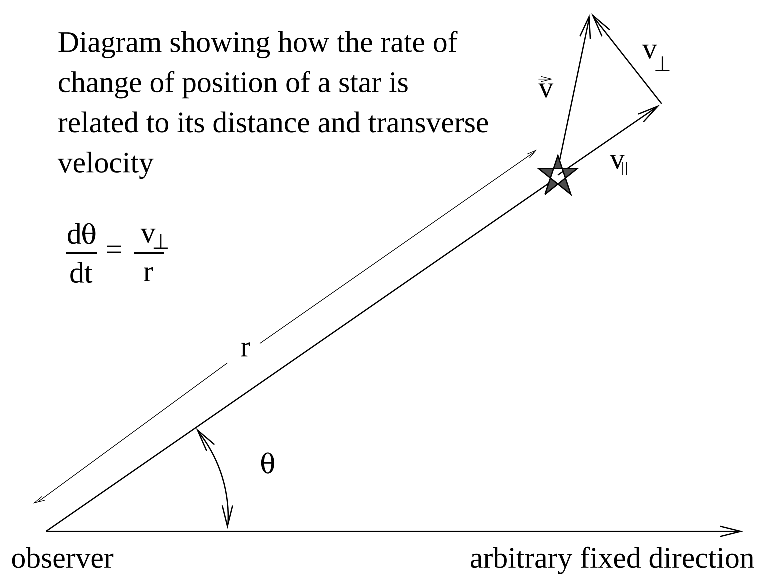
In the case of the earth, the coordinate motion which results from the torques of the other solar system bodies is called *precession of the equinoxes*. The name arises as follows.

can see that as the pole is dragged along, it carries the equator with it – the equator is always exactly 90 degrees to the pole. And this causes the intersection point of the equator and the ecliptic to be dragged along, so the equinoxes move gradually along the ecliptic due to precession.

Distant objects are in a fixed direction in space – the more distant the objects, the more slowly they appear to us to move. To see why this is so, imagine setting up polar coordinates (r, θ) with ourselves at the origin. Orient the plane containing the coordinates so that it contains the object, ourselves, and the object's velocity vector, as in the diagram below. The object will have some velocity parallel to the line of sight ($v_{\|}$) and another perpendicular to the line of sight ($v_{\perp}$). We have

$$v_{\perp} = r\frac{d\theta}{dt},$$

so if r is large, $d\theta/dt$ becomes very small. So distant objects maintain fixed directions in space.



Even so, their *equatorial coordinates* change gradually with time, because of precession of the equinoxes. *Therefore, whenever one quotes equatorial coordinates for an object, one must also quote the date of the coordinate system to which they refer.* This is sometimes loosely referred to as the *epoch* of the coordinates, though it's a little more correct to call it the *equinox* of the coordinates – most formally one sees 'referred to the mean equator and equinox of 1950', but one usually sees 'epoch 1950'.

Because the precession of the earth's axis is smooth, predictable, and well-understood, it's possible to convert coordinates from one epoch to another. To do this in a completely general fashion requires the use of the mathematics of rotation in three dimensions, which in turn requires the use of a matrix. But for approximate calculations, one can use a linear approximation – this simply takes the rate of change of the coordinates at some instant, and approximates the amount of change over a finite interval by multiplying the instantaneous rate of time by the interval. For precession, one has to a good approximation

$$\frac{d\alpha}{dt} = (3^{\mathrm{s}}.075 + 1^{\mathrm{s}}.336 \ \sin\alpha\tan\delta) \quad \mathrm{yr}^{-1},$$

and

$$\frac{d\delta}{dt} = 20''.043 \ \cos\alpha \quad \mathrm{yr}^{-1}.$$

Notice the units – for RA the expression gives the number of seconds of time per year, and for dec the number of seconds of arc per year.

For example, suppose one was given coordinates

$$\alpha = 13^{\mathrm{h}}\ 14^{\mathrm{m}}\ 15^{\mathrm{s}}.38, \quad \delta = -45^\circ\ 45'\ 45''.3, \quad \text{equinox 1950},$$

and asked to give coordinates for the same objects referred to the equinox 2000. One finds on converting that $\alpha = 13.237606$ hours, which is equivalent to 198.564083 degrees; converting δ to decimal degrees yields $\delta = -45.762583$. So we have

$$\sin\alpha = -0.3184,$$

$$\cos\alpha = -0.9480,$$

and

$$\tan\delta = -1.02698.$$

Putting these into the expressions yields

$$\frac{d\alpha}{dt} = (3^{\mathrm{s}}.075 + 0^{\mathrm{s}}.437) \quad \mathrm{yr}^{-1}$$

and

$$\frac{d\delta}{dt} = -19''.000 \quad \mathrm{yr}^{-1}$$

In 50 years this amounts to

$$\Delta\alpha = \left(\frac{d\alpha}{dt}\right)\Delta t = 175^{\mathrm{s}}.59,$$

and

$$\Delta\delta = \left(\frac{d\delta}{dt}\right)\Delta t = -950''.$$

Applying these to the original coordinates yields coordinates for the equinox of 2000

$$13^{\mathrm{h}}\ 14^{\mathrm{m}}\ 15^{\mathrm{s}}.38 \quad + \quad 175^{\mathrm{s}}.59 = 13^{\mathrm{h}}\ 17^{\mathrm{m}}\ 10^{\mathrm{s}}.98$$

and

$$-45^\circ\ 45'\ 45''.3 \quad - \quad 950''.0 = -46^\circ\ 01'\ 35''.3$$

A more accurate calculation using a full rotation-matrix formulation of the problem gives

$$\alpha = 13^{\mathrm{h}}\ 17^{\mathrm{m}}\ 11^{\mathrm{s}}.47, \quad \delta = -46^\circ\ 01'\ 33''.3, \quad \text{equinox 2000},$$

which shows that the approximations used are good but not perfect. Because of the $\tan\delta$ in the expression for $d\alpha/dt$, the errors clearly become largest near the poles; the errors also become larger the greater the time interval.

The computers attached to professional-class telescopes usually do precession calculations automatically, but one still must be careful. A large telescope typically has a very small field of view, often only a few arcminutes across. If one tells the telescope that one is aiming at coordinates for 2000, but feeds it 1950 coordinates by mistake, you can see that the error is substantial. In the example above, the error in RA would be 176 seconds of time, which is $176 \times 15 = 2640$ arcseconds; taking account of the fact that the lines of constant RA crowd together as one goes away from the equator, this is an angular distance along the small circle of $2640 \cos\delta = 1841$ arcsec. The miss distance in declination is 950 arcsec. RA and dec are perpendicular, and even though one is actually on the surface of a sphere, you can treat it *locally* as if it were flat (just as a surveyor treats a small piece of the spherical earth as a flat surface). So to find the total distance by which you'll miss the object, you add 1841 and 950 arcsec using the Pythagorean theorem,

$$\text{total miss distance} = \sqrt{1841^2 + 950^2} = 2072'',$$

which is $2072/60 = 34.5$ arcmin. So, if you use the wrong epoch, you generally can't find the object you're looking for at all!

More About Timekeeping

So far, we've mentioned *mean solar time* and *sidereal time*. Solar time is a strictly local quantity, because your meridian is different wherever you go on earth. This clearly won't do for keeping civilization organized, as early railroad companies discovered. Accordingly, the earth is divided into *time zones*, within which people agree to keep clocks set to the same value, which *approximates* the solar time. The time kept in a time zone is referred to as *zone time*; Eastern time is an example. In most regions of the world, zone times are offset from each other by an integer number of hours, but in some third-world countries smaller subdivisions are used. Time zones span on average 15 degrees of longitude, because one hour is equivalent to 15 degrees.

At any given moment, there is a time-zone boundary which is also the date boundary; on the west side of the line, the time might be 11:30 PM (23:30), and on the east side it is 12:30 AM (0:30) the next morning. When the hour switches, the date boundary moves one zone to the west. It keeps going around the earth over and over. Suppose this date boundary is the date between the first of the month and the second. Then it will remain the boundary between the first and the second, unless someone decides to re-set the date arbitrarily. Since we do want the date to advance from day to day, we use an *International Date Line*, near longitude 180 degrees, as an arbitrary point at which the new date begins.

When the date boundary reaches this point, the date is arbitrarily advanced by one; the new date then sweeps around the earth until the date line is reached again. Late in the day in California, you can call a person in Japan early the next morning, their time! Clearly, both date and time are dependent on your position on earth.

This is all hopelessly confusing and parochial, so astronomers and others who require unambiguous timing (e.g., the military) use *Universal Time (UT)*, which the military calls 'Zulu time'. Although we'll see later that the exact definition of UT is fraught with minor technicalities, the short story is that *UT is simply time for the time zone of Greenwich.* Eastern standard time is UT minus 5 hours, Central is UT minus 6 hours, Mountain is UT minus 7 hours, and Pacific is UT minus 8 hours. So when it is midnight UT, it is 5 PM Mountain.

Note carefully that UT also involves the *date*. When it is 7 PM Mountain time, on the 12th, it is 2 AM UT, *on the 13th*.

UT times are just about always recorded using the 24-hour clock, rather than AM and PM. So 17 hr UT is '5 PM' UT.

For civil timekeeping most locations in the US use *Daylight Savings Time* for part of the year. For this one simply advances the clock one hour in the spring ('spring forward') and brings it back to standard time in the autumn ('fall back'). When DST is in effect, the offsets to UT are 4 hr Eastern, 5 hr Central, 6 hr Mountain, and 7 hr Pacific – advancing the clock one hour places it one less hour behind Greenwich than it had been. The reason is that sunrise occurs very early in the summer months, before most people are up; by fiddling the clocks, one gets an extra hour of light in the afternoon, while still having light in the morning.

While UT rationalizes time recording considerably, it doesn't do anything about the calendar. Astronomers often want to find an accurate time interval between widely separated events. This is very difficult using the present calendrical system (quick! how many days old are you?). To make these calculations easier, you would like to have a time system which just consisted of a real number attached to each time, so you could just subtract two real numbers to get a time interval. There is such a system, called the *Julian Day system*. This is just a system of sequential day numbers, with the time of day expressed as the fractional part of the number. Julian day zero is around 4700 BC (they weren't invented then, it was just set up this way later). Therefore any historical record will have a positive Julian day number. The system was originally developed to record astronomical observations during the nighttime in Europe, so an unfortunate choice was made to have the Julian day change at *noon* UT; that way, all nighttime observations made in Europe would be on the same Julian date. With astronomy done world-wide, it's a major pain to have to get the extra 1/2 step correct. Julian day number 2 450 000 (two million, four hundred fifty thousand) was the same as UT date and time October 9, 1995, at 12 hours. Computer routines for converting standard calendar dates back and forth to Julian dates are available fairly readily.

More exact timekeeping depends on very careful definitions of time. Technically speaking, UT is based on earth rotation. Because all the civil clocks in the world are tied to

UT, you want the UT to keep pace with the mean sun, which in turn is based on the earth's rotation. But, as it turns out, the earth's rotation rate is not perfectly constant, but rather the length of the day gradually changes, by amounts measured in milliseconds per year. The general trend is for the earth to gradually slow down. A few milliseconds per year doesn't sound like much, but when you predict the angle to which the earth will be rotated in the future, the differences accumulate; a 5 millisecond change in the length of the day, persisting for a year, will accumulate to 0.005 sec $\times$ 365 days = 1.8 seconds; and persisting for 40 years it will accumulate to 73 seconds, which is extremely obvious!

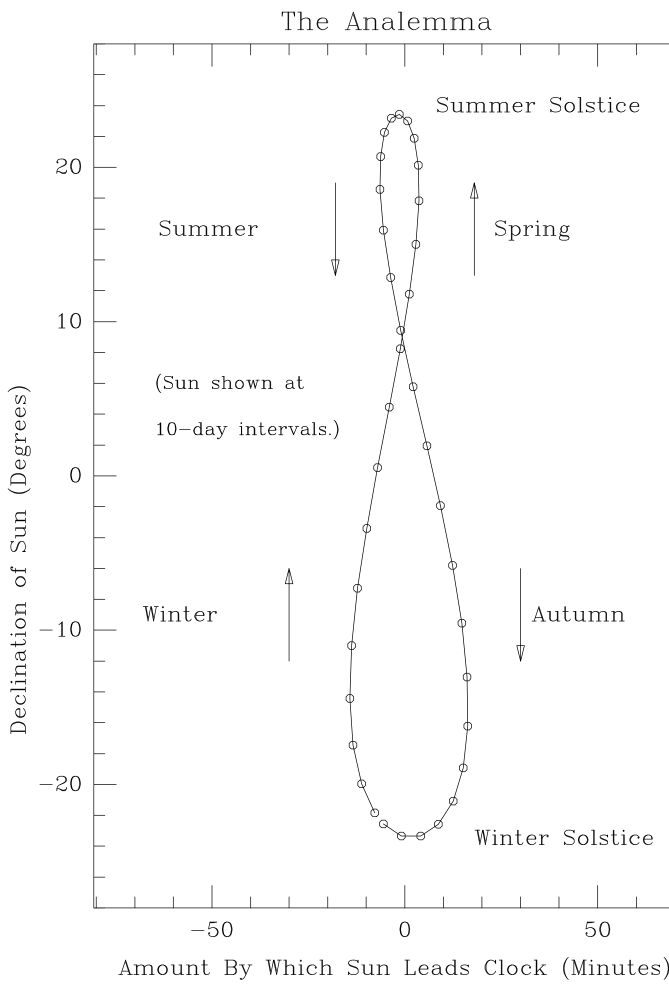
For this reason, it's necessary to define another timescale which is as uniform as we can make it. In the past, the only way to tell the earth was slowing down was to use the rest of the solar system as a 'clock'. The motion of the moon in particular, which is quite rapid, served as an independent time scale, and was used to define *Ephemeris Time* (ET), which used to be the most uniform timescale available. Then came atomic clocks, which are astonishingly accurate (with a stability approaching 1 part in 10^{14}!). This led to the definition of *International Atomic Time* (known by its French initials, TAI). TAI is independent of all astronomical observations. A new formulation of ET, called *Terrestrial Dynamical Time*, or TDT, is based on TAI and has superseded the old ET. For some reason the TAI scale is offset by a constant 32.184 seconds, in the sense that $\mathrm{TDT} - \mathrm{TAI} = 32.184$ sec.

Meanwhile, we still want a UT available for civil timekeeping. True UT – essentially the instantaneous phase of the rotation of the earth – is impossible to predict with perfect accuracy, because the fluctuations in the earth's rotation are not perfectly predictable. And if we did use true UT for timekeeping, the second would have to stretch and shrink according to the earth's rotation. So the international community uses instead a system called *Coordinated Universal Time*, or UTC, which is *close* to true UT, but differs from the very uniform TAI by an integer number of seconds. Because true UT does drift, an extra second is inserted from time to time to keep UTC within 0.9 seconds of true UT. These seconds, called *leap seconds*, are inserted in the last minute of December 31 or the last minute of June 30, as needed. So the last minute of December 31 or June 30 may have 61 seconds!

Returning now to the motions of bodies in the sky, you will recall that I made a distinction earlier between the mean sun (a fictitious body which moves at the sun's average east-west rate) and the real sun. The RA of the real sun does not move eastward at a constant rate. There are two reasons for this. First of all, the earth's orbit is not circular, so the motion of the sun along the ecliptic does not proceed at a uniform rate. Second, the ecliptic is inclined to the equator, so even if the sun were moving at a uniform rate, its right ascension would not increase at a constant rate. One can see this effect by imagining that the obliquity of the ecliptic were nearly 90 degrees. In that case the sun would be near 0 hours RA for half the year, then very rapidly switch to 12 hours RA! With a 23.4 degree obliquity, the effect is much less pronounced, but it's still there.

When the real sun is west of the mean sun (at a lower RA), it crosses the meridian earlier than the mean sun, and we say the sun is *fast*. When the real sun is east of the mean sun (higher RA), it crosses the meridian late, and we say it is *slow*. These effects can amount to nearly 1/2 hour. They are independent of your position on earth, and repeat consistently from year to year. The curve giving the amount by which the sun is fast or slow through the year is called the *equation of time*.

As we saw earlier, the sun's declination also varies systematically through the year, giving rise to the seasons. There is a lovely graph which plots the sun's declination on the vertical axis and the equation of time on the horizontal axis – this describes the path of the sun in the sky, after the earth's rotation has been taken out. The graph is called the *analemma*, and it is shown here.



Dennis DiCicco, a brilliant astrophotographer who is on the staff of the amateur astronomy magazine *Sky and Telescope*, once constructed a picture of the analemma by setting up a camera to snap a picture of the sun every morning at exactly the same time of day at 2-week intervals; the multiple exposure showed a huge figure-8 in the sky, just like the diagram. The analemma is sometimes plotted on a globe somewhere in the Pacific.

The asymmetry of the analemma can be used to explain a curious fact – though the longest night is on the solstice, the earliest sunset is a little before the solstice, around the

7th of December at mid-northern latitudes. Imagine repeating DiCicco's photograph with the camera pointing west, and set the time of day for your exposures to the moment of earliest sunset in December. The analemma will appear in the photograph tilted northward on the western horizon, and its lowest point will be tangent to the horizon. The unexpectedly early sunset is seen to be a consequence of the 'sun fast' condition before the solstice is reached.

Some Mathematical Techniques

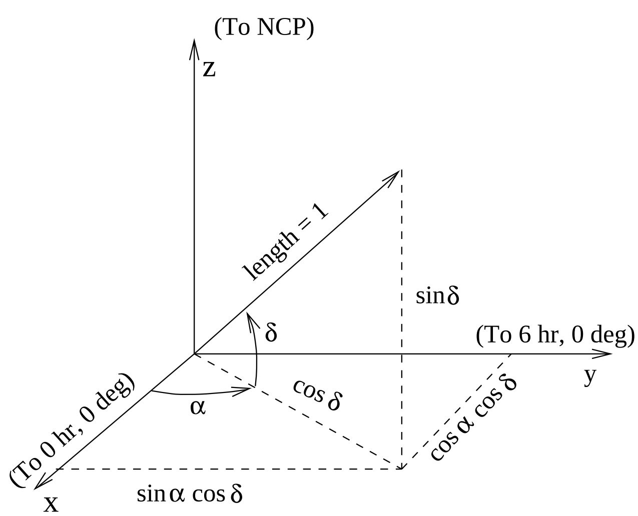
So far, we've mostly laid the conceptual groundwork for an understanding of this material. Now I would like to outline some of the mathematical techniques used for actually computing things. This is a large subject, which I will not cover completely, but it should be enough to let you do some practical problems.

It is often useful to transform spherical-polar coordinates to Cartesian *unit vectors*. These are simply vectors which have length 1, expressed as triplets of numbers; the first number is the x-coordinate, the second the y-coordinate, and the third is the z-coordinate. Refer to the diagram to verify that for right ascension α and declination δ,

$$\begin{aligned} z &= \sin\delta, \\ x &= \cos\delta\cos\alpha, \\ \text{and}\quad y &= \cos\delta\sin\alpha. \end{aligned}$$

By squaring these and summing these, you can see that these are components of a vector of length 1.

Transformation from RA and dec to (x,y,z) Coordinates



Notice that the x axis points towards $\alpha = 0, \delta = 0$, the y axis toward $\alpha = 6$ hr, $\delta = 0$, and the z axis toward $\delta = 90$ degrees. This is the standard configuration for Cartesian coordinates applied to celestial directions.

There are several things one can do with coordinates in this form. Suppose, for instance, that one has unit vectors $\mathbf{a}$ and $\mathbf{b}$. Then

$$\mathbf{a} \cdot \mathbf{b} = ab\cos\theta,$$

where a and b are the magnitudes, which are both one, and θ is the angle between the two vectors. Because these are unit vectors, a and b go out, leaving

$$\theta = \cos^{-1}(\mathbf{a} \cdot \mathbf{b}),$$

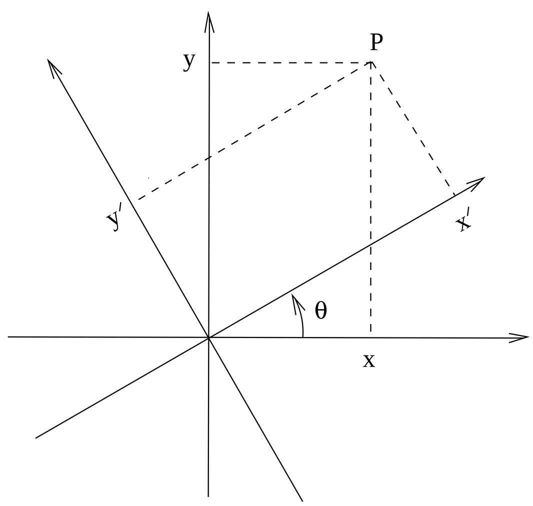
where $\cos^{-1}$ denotes the arc cosine function. So this is an easy way of finding the angle between any two directions.

Another useful feature of the vector form is the ease with which one can rotate coordinates. This is especially useful for precession, but the same method applies to other rotational transformations as well. I'll review this here. Suppose one has coordinates x and y in two dimensions, and you wish to rotate *the axes* counter-clockwise through angle θ, to make new axes x' and y'. Then the coordinates of the same point referred to the rotated axes are

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The 2×2 matrix is called a *rotation matrix*. Notice that it reduces to the identity matrix when $\theta = 0$, and has a determinant of one.

Rotation of Coordinate Axes



You can extend this to three dimensions simply by rotating around each axis in turn. This gives you the freedom to do a completely general rotation. To extend the formalism above to rotations around any of the three axes, you stretch the matrix out to 3×3, and pad with ones and zeros as appropriate:

$$\text{rotation around } x \text{ axis}: \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

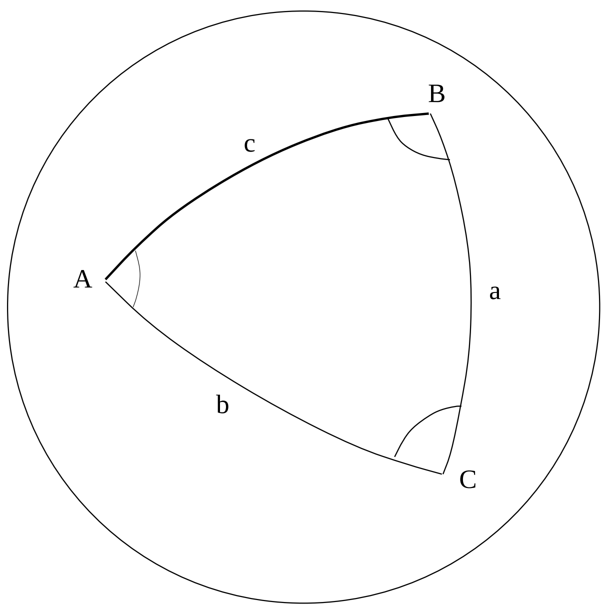
$$\text{rotation around } y \text{ axis}: \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}$$

$$\text{rotation around } z \text{ axis}: \quad \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

To get the fully general rotation, you multiply these all together. Since matrix multiplication is not commutative, the ordering of these matrices it *does* matter. When you write out the matrices for multiplication, the first rotation to be applied is the rightmost, the second is the middle, and the last is the leftmost. Matrix multiplication is associative, so it doesn't matter how you group the multiplication once you've written them down.

An entirely different, and much older, approach to practical astronomy problems is the use of the *spherical triangle*. This is a triangle, all of whose arcs are sections of *great circles* (not small circles!). It's customary to label the angles of the three vertices as A, B, and C, and the arcs *opposite* these three vertices as a, b, and c. The sphere is understood to be a unit sphere, so the arcs can serve as angles – it makes sense to speak of $\sin a$, even though a is an arc.

A spherical Triangle



a, b, and c are all sections of great circles.

It's interesting to note that on a sphere, the sum of the angles $A + B + C > 180$ degrees; it only approaches 180 degrees as the triangle becomes small compared to the sphere's radius.

Spherical triangles have their own trigonometry. Two results are especially useful, namely the *spherical law of sines*,

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C},$$

and the *spherical law of cosines*, which takes two forms

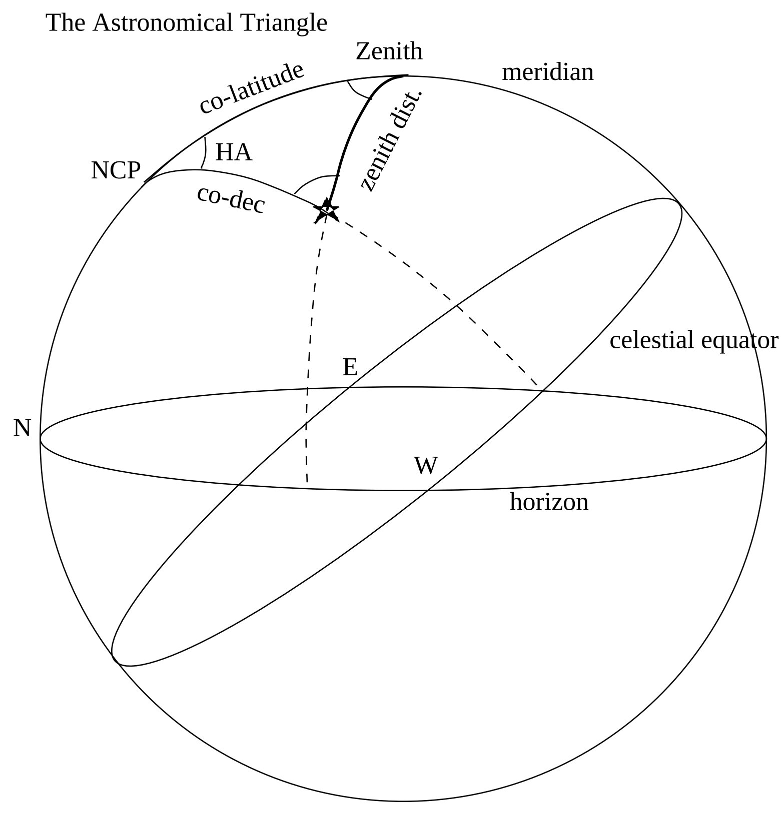
$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

and

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a.$$

With the first of these, we can prove a little lemma which will be useful, namely that if two sides b and c of a spherical triangle are 90 degrees long, the angle between them A is equal to the arc a. This is pretty obvious intuitively – imagine a triangle formed by the north pole and two points on the equator. Then the length of the arc along the equator is just equal to the angle subtended at the north pole, which will just be the difference in the longitude of the two points.

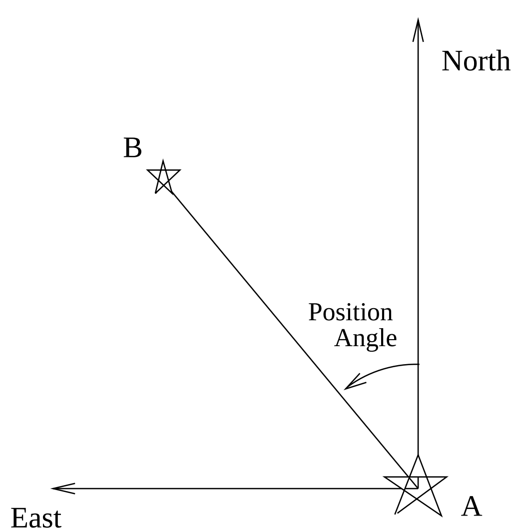
While spherical trigonometry is very useful in problems involving the celestial sphere, it is perhaps most commonly applied to the following triangle, which is sometimes called the *astronomical triangle*. This is a triangle on the celestial sphere whose vertices are the object you're observing, the north celestial pole, and the zenith. The arc between the



zenith and the north celestial pole will equal your co-latitude, and the arc between the NCP and your object will be the object's co-declination, or 90 degrees minus its dec. A little less obvious is the fact that the angle of the vertex at the NCP is the object's hour angle (which follows from our little lemma applied to the triangle whose vertices are the NCP, the meridian at the equator, and the hour angle of the object at the equator). Putting this together we have 'side-angle-side', which completely specifies the triangle. One can now use the laws of spherical trigonometry to find the object's *zenith distance*, which is the angular distance from the zenith, and the object's *azimuth*. The azimuth is the 'left-right' position of the object; it's a longitude-like coordinate in the spherical polar system with a 'pole' at your zenith and an 'equator' coincident with your horizon. The object's altitude, which we introduced earlier, will be 90 degrees minus its zenith distance.

There's even more we can get! Once we've found the zenith distance and the azimuth, we have all the elements of the triangle except for the angle at the vertex near the object. This angle is a useful quantity; to see what it's used for, we introduce the idea of *position angle*. Imagine looking at an object in the sky and taking its picture. The picture you're looking at will have a well-defined northerly direction, which is the direction of an arc toward the NCP. This will not in general be coincident with up and down, or with left and right. For example, if you look toward the eastern horizon, north will be upward and toward the left. Now that the northerly direction in a picture is defined, imagine trying to describe how one object lies with respect to another – for example, the direction between the fainter component of a double star and the brighter one. You describe this using the position angle, which is the angle the arc from one star to the other makes to the north. This is pictured here. It is puzzling at first to note that if north is to the top, east is to the left in an astronomical picture; that's because we're inside the celestial sphere, looking out. By contrast, a conventional map of terrain on earth is from the outside looking down (a bird's-eye view), which has the opposite parity (or handedness).

Position Angle of star B around star A



Measured "North through East"

(East is to the left in celestial maps)

The last angle in the astronomical triangle is the position angle of the arc toward the zenith – the position angle of 'straight up'. This is called the *parallactic angle*, because topocentric parallax (covered later) works along this angle. It's important to optical observations because the refraction in the earth's atmosphere tends to make objects appear a little higher in the sky than they really are, so this effect moves objects along the parallactic angle.

So you can see that spherical triangles will be useful, especially when one needs to know the angle with which two great-circle arcs intercept. For that problem, the vector formulations are not useful.

The Moon

The moon orbits the earth at a mean distance of about 60.3 earth radii. Its average angular diameter is about 31 arcmin, corresponding to a physical radius of 1738 km, about 1/4 that of the earth.

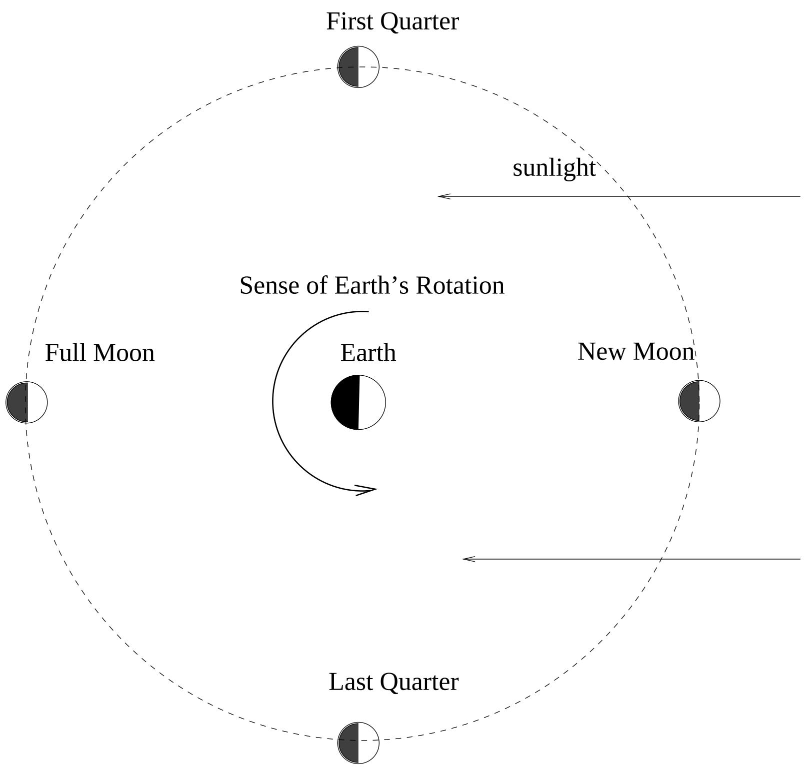
The moon's orbit is approximately elliptical, as one would expect from the solution of the gravitational two-body problem. The eccentricity of the moon's orbit is about 0.055. It takes on average 27.32 days for the sun to return to the same RA in the sky – this is called the *sidereal month*. However, during this time the sun has moved about 2 hr in RA, so it takes a couple of days for the moon to come back to the same phase with respect to the sun. As I would hope anyone would know, the moon's obvious phases are due to our changing view of the moon's illuminated face, so the time required for the moon to go through its phases is controlled by how long it takes to come back to the same phase with respect to the sun. This is the *synodic month*, which is 29.53 days. Note that

$$\frac{1}{1 \text{ sidereal month}} - \frac{1}{1 \text{ synodic month}} = \frac{1}{1 \text{ year}},$$

so the difference between the synodic and sidereal months is precisely analogous to the difference between the solar and sidereal days.

To explore the phases of the moon more fully it's useful to define *ecliptic coordinates*, in particular *ecliptic longitude* and *ecliptic latitude*. These are spherical-polar coordinates just like RA and dec, or geographical latitude and longitude. Ecliptic latitude is measured away from the ecliptic, and ecliptic longitude is measured along the ecliptic eastward from the first point of Aries. So ecliptic coordinates are broadly similar to RA and dec, but with the pole tilted by 23.4 degrees away from the earth's pole.

When the moon's ecliptic longitude is the same as the sun's we say the moon is *new*. It is then invisible, unless it is silhouetted against the sun in an eclipse. Because the moon's orbit is inclined somewhat to the ecliptic, this does not always happen (more later). About a week later, the moon lies at an ecliptic longitude 90 degrees away from that of the sun (for some reason, hours are not generally used for ecliptic longitude as they are for RA), and we say the moon is at *first quarter*. It's called quarter because 1/4 of the cycle has passed; it actually appears as a *half* moon. The first quarter moon stands near the meridian at sunset, and sets sometime around midnight. About a week later, the moon's longitude is exactly 180 degrees from the sun, and the moon is *full*. It now rises at sunset, transits at midnight, and sets at sunrise. The full moon is very bright, so the whole night is illuminated. About a week later, when the moon is 270 degrees of ecliptic longitude from the sun, we have *last quarter*; the last quarter moon rises around midnight, transits around sunrise, and sets around noon. Then a week later we have new moon again, and the cycle begins again. Each of these phase cycles is called a *lunation*. The diagram shows (schematically) the phases of the moon, as viewed from high above the north pole; notice that the sense of the moon's revolution is the same as that of the rotation of the earth. If you are unfamiliar with this it is a good exercise to carefully verify the statements regarding where the moon can be found in the sky at various times of day.



The moon's orbit is inclined to the ecliptic by about 5.1 degrees. Therefore, it moves more-or-less along the plane of the ecliptic. Therefore, the declination of the full moon is about opposite to that of the sun. In winter, the full moon is high in the sky; in summer, it is low on the horizon. Notice that this applies only to the *full* moon.

The moon crosses the ecliptic twice per month. If this happens to correspond to the time the sun is at the same longitude, we have an *eclipse* – that's where the term 'ecliptic' comes from! If the moon moves across the sun at new moon, one has a *solar eclipse*, in which the moon casts a shadow on the earth; if the moon moves across the ecliptic at full moon, the earth will cast a shadow on the moon, and we have a *lunar eclipse*. We'll discuss these a little more thoroughly in a moment.

The moon's orbit is strongly affected by the sun – in fact, the sun's gravity exerts a stronger pull on the moon than does the earth's gravity, so the moon's path is always concave toward the sun, even when the earth is pulling it the other way. But because the earth and the moon orbit the sun together, we see the moon as going around us. Still, the gravity of the sun (and to some extent of the planets) causes very large *perturbations* on the moon's orbit. There are two major ones:

- The *line of nodes* where the plane of the moon's orbit crosses the ecliptic precesses toward smaller longitudes with a period of 18.61 years; and
- the longitude of the moon's perigee (the closest point in its orbit to earth) rotates toward higher longitudes with a period of 8.85 years.

An accurate accounting of the moon's orbit requires hundreds of perturbation terms; the most accurate lunar *ephemerides* (a term for predictions of the positions of celestial bodies, pronounced 'eff-emm-air-id-ees') now come from computer integrations of the equations of motion.

It's worth noting again (as we did on the first page) that the moon is so close to the earth that its apparent position is strongly affected by your position on earth. This effect is called *parallax*, which is a general term for changes in apparent position due to changes in viewing point; parallax is greatest for nearby objects. Parallax effects due to viewing positions on earth are called *topocentric parallax*, because positions referred to an observer's geographical position are called *topocentric*. By contrast, positions referred to a hypothetical observer at the center of the earth (who is somehow able to see out) are called *geocentric* positions. The correction from geocentric to topocentric position for the moon amounts to almost one degree when the moon is on the horizon; it's zero when the moon is in the zenith.

Other Coordinate Systems

Here I'd like to collect together a few of the other coordinate systems used in astronomy. I've alluded to most of them already, but I should lay them out more systematically. The equatorial system has been covered exhaustively earlier. All of these are coordinate systems on the celestial sphere – they describe direction only.

Ecliptic Latitude and Longitude. These have been covered earlier. Briefly, the pole is the direction perpendicular to the ecliptic, and the zero point of longitude is the first point of Aries. Ecliptic latitude and longitude are generally given in degrees.

Altitude and Azimuth. These are sometimes called *topocentric* coordinates – they are strictly local to an observer on earth. *Altitude* is the angle between the point in question and the observer's horizon, measured along a great circle which also passes through the zenith. *Zenith distance* is the complement of altitude. *Azimuth* is a longitude-like coordinate measured along the horizon, starting at due north and proceeding through east to the point at which a great circle through the zenith and the object intercepts the horizon. Thus an object due east has an azimuth of 90 degrees, due south has 180 degrees, and due west has 270 degrees.

Galactic coordinates are referred to a pole which is perpendicular to the plane of the Milky Way. Galactic latitude is called b and Galactic longitude is called l. The zero of galactic longitude is roughly coincident with the direction toward the center of the Galaxy, and increases roughly eastward. Galactic longitudes are defined from 0 to 360 degrees, rather than in ±180 degrees, which is kinda dumb. The plane of the solar system has nothing to do with the plane of the Galaxy – they're randomly oriented with respect to each other – so rotating from one to the other calls for a full 3-d coordinate transformation. Recall that all these coordinates are on the celestial sphere, so it doesn't matter where the center of galactic coordinates is taken to be – the galactic coordinates refer only to directions in space. The center and pole of the galaxy (referred to equinox 1950) are approximately

$$\text{center:}\ \ \alpha = 17^{\rm h}\ 42^{\rm m}.4,\ \ \delta = -28^\circ\ 55',$$

$$\text{pole:}\ \ \alpha = 12^{\rm h}\ 49^{\rm m}.0,\ \ \delta = +27^\circ\ 40',$$

An older system used before the early 1960s had a different zero of longitude, but that's all long gone now.

Some Finer Details

There are several small effects on a star's position which I've ignored so far. They're important for precise work, and you'll see them mentioned in other sources.

- *Nutation* is a small variation of RA and dec – less than 1 arcmin – caused by slight wobbles of the direction of the earth's axis. These are rather complicated – to compute them one generally uses an extensive series of terms. Nutation is superposed on the smooth variation of precession. One often sees coordinates referred to the *mean equinox*, which means that the effects of nutation have been ignored in the calculation. One also sees a distinction between *local mean sidereal time* and *local apparent sidereal time*, which arises as follows. The sidereal time is the hour angle of the first point of Aries, or the vernal equinox. Because the vernal equinox is the point where the equator crosses the ecliptic, a wobble in the direction of the pole causes the vernal equinox to wobble slightly, too. Local *apparent* sidereal time is the hour angle of the true equinox; local *mean* sidereal time is the hour angle of the *mean* equinox, for which nutation is ignored. So local mean sidereal time is a somewhat more regular timescale than local apparent sidereal time. Nearly all coordinates quoted in the literature are referred to the mean equinox.

- *Aberration* is a slight change of apparent position caused by the earth's motion and the finite speed of light. The earth moves about 30 km/s in its orbit, which is 10^{-4} of the speed of light; accordingly, a star which lies perpendicular to the direction of the earth's motion will have its position shifted by about 10^{-4} of a radian, which amounts to some 20 seconds of arc. The effect is to make the star appear slightly closer to the direction toward which the earth is moving.

- *Refraction* is the displacement of a star's image by the earth's atmosphere. For a star 60 degrees from the zenith, this amounts to about 1.6 arcmin at sea level. The effect is to raise the star's image above where it would have been without an atmosphere. It's good to remember that refraction is not independent of wavelength – the blue image of a star is refracted somewhat more than the red image, so stars near the horizon look like little rainbows under magnification.

- *Proper motion* is a change in the apparent position of a star due to its actual physical motion across the line of sight. The largest proper motion (Barnard's star) is about 10 arcsec per year. Proper motion is only large for very nearby stars, though it does accumulate to a noticeable displacement over time even for more distant stars. Extragalactic objects generally have proper motions indistinguishable from zero.

- *Annual Parallax* is the shift in the apparent position of a nearby star due to the motion of a the earth in its orbit, which causes the viewing point to shift. The nearest star has an annual parallax of only about 0.7 arcsec, and more distant stars all have parallaxes smaller than this. Parallax for stars is usually so small that it's hard to detect and measure accurately. It never causes large displacements in the positions of stars.

The RA and dec of a star including nutation, aberration, proper motion, and annual parallax – is often called the *apparent place* of a star. If one really needs to point a telescope exactly at a star – as with a large professional telescope – one needs to compute the apparent place first, and then account for refraction and any known errors in the telescope mounting.

There's also a minor distinction in the definition of *epochs*, which are just moments in time to which astronomical positions are referred (for precession, etc.). The standard practice nowadays is to refer to *Julian epochs*, which are measured in years of 365.25 days from the standard epoch denoted *J2000*, which is Julian day 2451545. exactly. This corresponds to 12 hours UT on 2000 Jan 1. An older practice was to use *Besselian epochs*, in which the length of the year is taken to be 365.2422 days and the fundamental epoch is *B1950*, which is JD 2433282.423, corresponding to 1949 December 31 at 22:09:07 UT.

Some Topics Left Out . . .

I'm not going to treat the motion of the planets in any detail here, save to say that they orbit the sun approximately in the plane of the ecliptic. There are some very obvious inferences about the planets, such as the fact that the inferior planets (Venus and Mercury) never reach large angular distances from the sun; they are never visible at midnight from temperate latitudes. Planetary motions are well described in other books.

Further Reading

The 'Bible' for calculations of this kind is the *Astronomical Almanac.* This book is published annually by the U.S. Government Printing Office and Her Majesty's Stationery Office. It has definitive tables of planetary positions, and formulae for transforming coordinates and timescales. It's available in any good scientific library.

Unsöld and Baschek's wonderful little book *New Cosmos* (Springer Verlag, 1991) contains a short section which summarizes much of the information given here. It's considerably less verbose than my treatment, which will appeal to many readers.

L. G. Taff's *Computational Spherical Astronomy* (Wiley, 1981) is another useful reference.

Finally, the Belgian amateur Jean Meeus has written at least two very useful cookbooks for calculations of this kind, the more useful of which is *Astronomical Formulae for Calculators* (Willman-Bell). This contains a rather accurate lunar theory, among other things.

The popular magazine *Sky and Telescope* has discussions of issues of this kind from time to time, and is a rich source for advertisements of relevant personal computer software. I've been very pleased with the inexpensive *Guide* package for IBM PC clones, available from Project Pluto in Bowdoinham, Maine.

Finally, users with access to workstations and Internet access might want to explore my own software, *skycal*, which is available for free via anonymous ftp from iraf.noao.edu, in the contrib directory. This package contains a large number of c-language routines to do this kind of calculation, and a manual is included as well.

Please send comments to john.thorstensen@dartmouth.edu .